THE THEORY

AND

PRACTICE OF MECHANICS

BY

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PREFACE

A COURSE in theoretical mechanics often lacks the strength and vitality resulting from practical applications, whereas a course in applied mechanics is likely to be so technical and limited in its range as to obscure the meaning and scope of the fundamental principles involved. By properly combining theory and practice the efficiency of instruction is greatly increased, and mechanics also becomes a powerful instrument for coördinating mathematics and physics with technology.

In this text the aim is to present the fundamental principles of mechanics in such a way as to emphasize their actual significance and relationship, and at the same time make them a matter of intelligent interest to the average student of junior grade in colleges and universities, and in technical and engineering schools. In each article the explanation is given a body by direct application to some practical problem within the range of the student's experience or understanding.

Much of the subject matter will be found useful in reviewing earlier work in related subjects, thus unifying and correlating the student's knowledge, and breaking down the artificial boundaries between "pure" mathematics, "theoretical" physics, and "applied" mechanics. For instance, the subject of elastic vibrations, presented in Articles 105 and 106, involves a physical interpretation of Maclaurin's theorem, simple trigonometric relations, the graphical representation of trigonometric functions, elementary differential equations, simple integration, the definition and properties of harmonic motion, the composition of waves, the formation of nodes in vibrating strings, and practical engineering applications, although the discussion is so simple as to be easily intelligible. This aim has been kept in mind throughout the preparation of the text.

The writer also believes that a student's college text in mechanics should find a permanent place in his professional library, as he will feel greater confidence in consulting a book with which

he has become familiar during his undergraduate work than one entirely new. For the sake of completeness, therefore, many things have been included which it may be found expedient to omit in an elementary course. For a brief course it is suggested that the first three chapters may prove sufficient, as they cover the ground usually presented in an elementary text. This course may be extended by adding Chapters IV and VII for students in mechanical engineering, or Chapters V and VI for students in civil and electrical engineering.

An essential feature of a satisfactory text is a consistent and logical notation, although this is extremely difficult to secure where the subject involves more concepts than there are symbols available. The best practice when possible is to use the initial letter of the name of a quantity as its symbol, such as the universal use of g for acceleration due to gravity. Some of the symbols here used as an aid to the memory are M for mass, T for torque, P for period, F for force, A for area, I for moment of inertia, etc. Where two quantities with the same symbol occur together, as time and torque, or impulsive couple, in the discussion of angular momentum (Art. 164 *et seq.*), the letter M is used for the moment of the impulsive couple to distinguish between it and the time t, since the symbol M is not needed for mass in this connection, as only the moment of inertia I is involved.

A feature of the book is the collection of 420 practical problems, the majority of which are original.

Acknowledgment is due the following firms and organizations for the use of cuts and half-tones from their trade catalogues:

American-Ball Engine Co., Figs. 263, 280, 283; American Hoist and Derrick Co., Figs. 126, 127, 142; American Locomotive Co., Figs. 281, 282, 291, 292, 293, 294; Ahrens-Fox Fire Engine Co., Fig. 286; Cincinnati Milling Machine Co., Fig. 79; Dodge Mfg. Co., Figs. 190, 191; Esterline Co., Fig. 82; Hess-Bright Mfg. Co., Figs. 182, 183; Johnston Harvester Co., Fig. 109; McClure's Magazine, Fig. 312; Rockwood Mfg. Co., Figs. 186, 188; Timkin Roller Bearing Co., Fig. 185; Treasury Department, U. S. Government, Figs. 87, 88.

S. E. S.

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January, 1913.

CONTENTS

CHAPTER I

KINEMATICS

CHAPTER II

FUNDAMENTAL DYNAMICAL PRINCIPLES

CHAPTER III

STATICS

CHAPTER IV

FRICTION AND LUBRICATION

CHAPTER V

KINETICS OF PARTICLES

CHAPTER VI

KINETICS OF RIGID BODIES

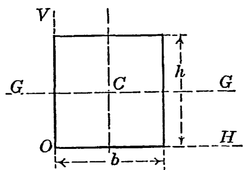
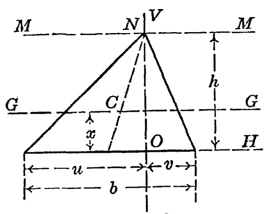
CHAPTER VII

DYNAMICS OF ROTATION

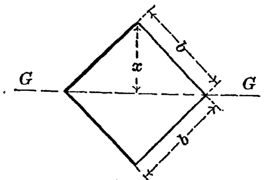
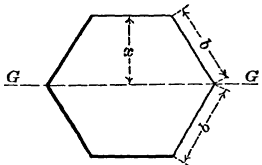
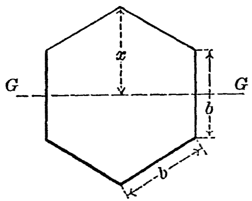
TABLES OF MATHEMATICAL AND PHYSICAL CONSTANTS

TABLE 1

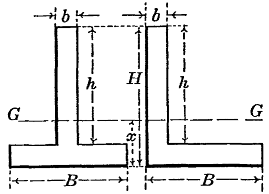
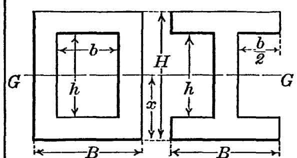
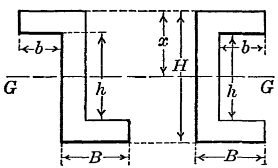
Properties of Various Plane Sections

Shape	Area	Location of Gravity Axis	Moment of Inertia
 *Rectangle*	bh	Center	$I_{OH} = \frac{bh^3}{3}$ $I_{OV} = \frac{hb^3}{3}$ $I_{GG} = \frac{bh^3}{12}$ $I_C\,(\text{polar}) = \frac{bh}{12}(b^2 + h^2)$
 *Triangle*	$\frac{1}{2}\,bh$	$x = \frac{h}{3}$	$I_{OH} = \frac{bh^3}{12},\ I_{GG} = \frac{bh^3}{36},\ I_{MM} = \frac{bh^3}{4}$ $I_C(\text{polar}) = \frac{bh^3}{36} + \frac{h}{12}(u^3 + v^3)$ $- \frac{bh}{18}[2(u^2 + v^2) - b^2]$ $I_N(\text{polar}) = \frac{bh^3}{4} + \frac{h}{12}(u^3 + v^3)$

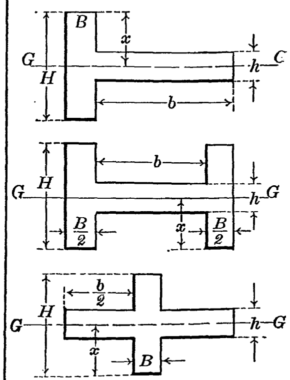
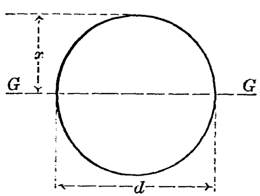
PROPERTIES OF VARIOUS PLANE SECTIONS — *Continued*

SHAPE	AREA	LOCATION OF GRAVITY AXIS	MOMENT OF INERTIA
 *Square*	b^2	$x = \frac{b}{\sqrt{2}} = .707\,b$	$I_{GG} = \frac{b^4}{12}$
 *Hexagon*	$\frac{3\sqrt{3}}{2}b^2 = 2.6\,b^2$	$x = \frac{\sqrt{3}}{2}b = .866\,b$	$I_{GG} = \frac{5\sqrt{3}}{16}b^4 = .5413\,b^4$
 *Hexagon*	$\frac{3\sqrt{3}}{2}b^2 = 2.6\,b^2$	$x = b$	$I_{GG} = \frac{5\sqrt{3}}{16}b^4 = .5413\,b^4$

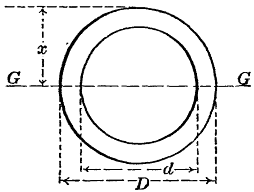
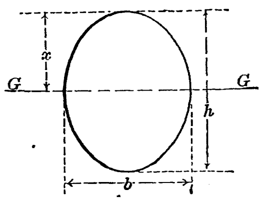
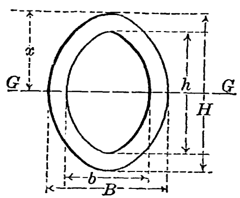
PROPERTIES OF VARIOUS PLANE SECTIONS — *Continued*

SHAPE	AREA	LOCATION OF GRAVITY AXIS	MOMENT OF INERTIA
	$B(H-h)+bh$	$x=\dfrac{BH^2-h(B-b)(2H-h)}{2[B(H-h)+bh]}$	$I_{GG}=$ $\dfrac{b(H-x)^3+Bx^3-(B-b)(x+h-H)^3}{3}$
 	$BH-bh$	$x=\dfrac{H}{2}$	$I_{GG}=\dfrac{BH^3-bh^3}{12}$

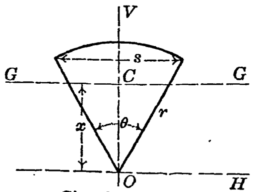
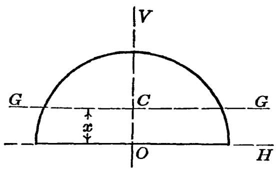
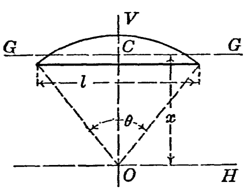
PROPERTIES OF VARIOUS PLANE SECTIONS — *Continued*

SHAPE	AREA	LOCATION OF GRAVITY AXIS	MOMENT OF INERTIA
	$BH + bh$	$x = \frac{H}{2}$	$I_{GG} = \frac{BH^3 + bh^3}{12}$
	$\frac{\pi d^2}{4} = .785\, d^2$	$x = \frac{d}{2}$	$\frac{\pi d^4}{64} = .049\, d^4$

PROPERTIES OF VARIOUS PLANE SECTIONS — *Continued*

SHAPE	AREA	LOCATION OF GRAVITY AXIS	MOMENT OF INERTIA
	$\frac{\pi(D^2-d^2)}{4}=.785(D^2-d^2)$	$x=\frac{D}{2}$	$\frac{\pi(D^4-d^4)}{64}=.049(D^4-d^4)$
	$\frac{\pi bh}{4}=.785\,bh$	$x=\frac{h}{2}$	$\frac{\pi bh^3}{64}=.049\,bh^3$
	$\frac{\pi}{4}(BH-bh)=.785(BH-bh)$	$x=\frac{H}{2}$	$\frac{\pi(BH^3-bh^3)}{64}=.049(BH^3-bh^3)$

PROPERTIES OF VARIOUS PLANE SECTIONS — *Continued*

SHAPE	AREA	LOCATION OF GRAVITY AXIS	MOMENT OF INERTIA
 *Circular Sector*	$\frac{r^2\theta}{2}$	$x = \frac{4r}{3}\frac{\sin\frac{\theta}{2}}{\theta}$ $= \frac{sr^2}{3A}$ $(A = \text{area})$	$I_O\ (\text{polar}) = \frac{r^4\theta}{4}$ $I_C\ (\text{polar}) = \frac{r^4}{4}\left(\theta - \frac{1-\cos\theta}{\theta}\right)$
 *Semicircle*	$\frac{\pi r^2}{2}$	$x = \frac{4r}{3\pi}$	$I_{OH} = I_{OV} = \frac{\pi r^4}{8}$ $I_O\ (\text{polar}) = \frac{\pi r^4}{4}$ $I_C\ (\text{polar}) = \frac{r^4}{4}\left(\pi - \frac{2}{\pi}\right)$ $= .726\,r^4$
 *Circular Segment*	$\frac{r^2}{2}(\theta - \sin\theta)$	$x = \frac{4r}{3}\frac{\sin\frac{\theta}{2}}{\theta - \sin\theta}$ $= \frac{l^3}{12A}$ $(A = \text{area})$	$I_O\ (\text{polar}) = \frac{r^4}{4}(\theta - 2\sin\theta\cos^3\theta$ $- \frac{2}{3}\cos\theta\sin^3\theta)$ $= \frac{r^4\theta}{4} - \frac{1}{4}\left[lr^3\cos^3\theta + \frac{rl^3\cos\theta}{12}\right]$

PROPERTIES OF VARIOUS PLANE SECTIONS — *Continued*

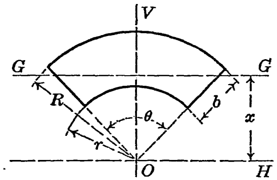
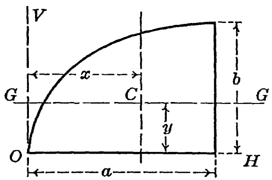
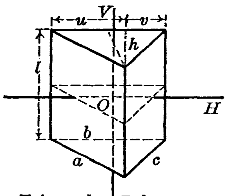
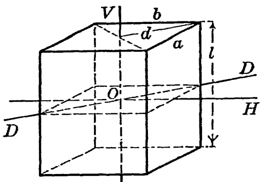
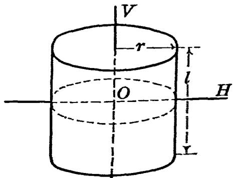
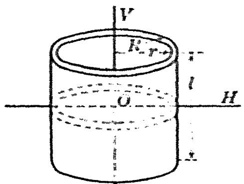
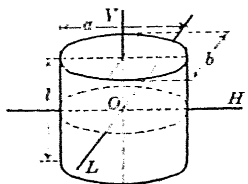
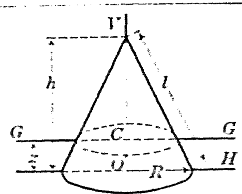
SHAPE	AREA	LOCATION OF GRAVITY AXIS	MOMENT OF INERTIA
 *Segment of Annulus*	$\frac{\theta}{2}(R^2 - r^2)$	$x = \frac{4}{3} \cdot \frac{R^3 - r^3}{R^2 - r^2} \cdot \frac{\sin \frac{\theta}{2}}{\theta}$	$I_O \text{ (polar)} = \frac{\theta}{4}(R^4 - r^4)$
 *Semi Parabola*	$\frac{2}{3} ab$	$x = \frac{3}{5} a$ $y = \frac{3}{8} b$	$I_{OH} = \frac{8}{15} ab^3$ $I_{OV} = \frac{32}{105} ba^3$

TABLE 2

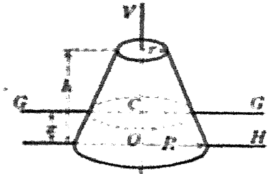
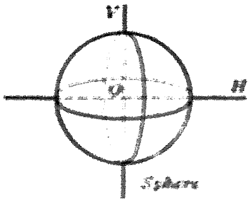
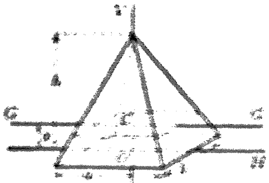
Properties of Various Solids

Shape	Surface	Volume and Mass	Center of Gravity	Moment of Inertia
 Triangular Prism	Sides $A = l(a+b+c)$ End $A = \frac{1}{2}bh$	$V = \frac{1}{2}bhl$ $M = \frac{1}{2}bhl\delta$ (δ = density)	Center of Figure	$I_{OH} = \left(\frac{bhl^3}{24} + \frac{blh^3}{36}\right)\delta$ $= M\left(\frac{l^2}{24} + \frac{h^2}{36}\right)$ $I_{OV} = M\left[\frac{bh^3}{18} + \frac{h}{6}(u^3+v^3) - \frac{bh}{9}\{2(u^2+v^2) - b^2\}\right]$
 Rectangular Prism	Sides $A = 2l(a+b)$ End $A = ab$	$V = abl$ $M = abl\delta$	Center of Figure	$I_{OH} = M\left(\frac{l^2+a^2}{12}\right)$ $I_{OV} = M\left(\frac{a^2+b^2}{12}\right)$ $I_{DD} = M\left(\frac{l^2}{12} + \frac{d^2}{6}\right)$
 Circular Cylinder	Lateral Surface $A = 2\pi rl$ End $A = \pi r^2$	$V = \pi r^2 l$ $M = \pi r^2 l\delta$	Center of Figure	$I_{OH} = M\left(\frac{l^2}{12} + \frac{r^2}{4}\right)$ $I_{OV} = \frac{1}{2}Mr^2$

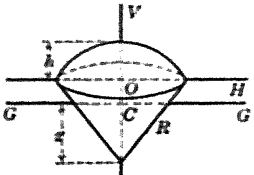
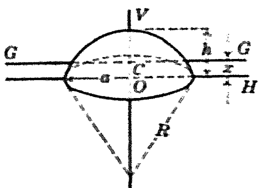
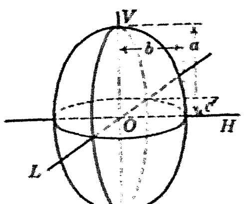
PROPERTIES OF VARIOUS SOLIDS — *Continued*

SHAPE	SURFACE	VOLUME AND MASS	CENTER OF GRAVITY	MOMENT OF INERTIA
 *Hollow Cylinder*	Lateral Surface $A = 2\pi l(R + r)$ End $A = \pi(R^2 - r^2)$	$V = \pi l(R^2 - r^2)$ $M = \pi l(R^2 - r^2)\delta$	Center of Figure	$I_{OH} = M\left[\frac{l^2}{12} + \frac{R^2 + r^2}{4}\right]$ $I_{OV} = M\left(\frac{R^2 + r^2}{2}\right)$
 *Elliptic Cylinder*	End $A = \frac{\pi ab}{4}$	$V = \frac{\pi abl}{4}$ $M = \frac{\pi abl\delta}{4}$	Center of Figure	$I_{OH} = M\left(\frac{l^2}{12} + \frac{b^2}{16}\right)$ $I_{OL} = M\left(\frac{l^2}{12} + \frac{a^2}{16}\right)$ $I_{OV} = M\left(\frac{a^2 + b^2}{16}\right)$
 *Circular Cone*	Lateral Surface $A = \pi R\sqrt{h^2 + R^2}$ Base $A = \pi R^2$	$V = \frac{1}{3}\pi R^2 h$ $M = \frac{1}{3}\pi R^2 h\delta$	$x = \frac{h}{4}$ [For lateral surface only $x = \frac{h}{3}$]	$I_{GG} = \frac{3}{20} M\left(R^2 + \frac{h^2}{4}\right)$ $I_{OV} = \frac{3}{10} MR^2$

Properties of Various Solids — *Continued*

Shape	Surface	Volume and Mass	Center of Gravity	Moment of Inertia
 Frustrum of Cone	Lateral Surface $A = \pi(R+r)\sqrt{h^2+(R-r)^2}$ Base $A = \pi R^2$ Top $A = \pi r^2$	$V = \frac{\pi h}{3}(R^2+Rr+r^2)$ $M = \frac{\pi h \delta}{3}(R^2+Rr+r^2)$	$x = \frac{h}{4}\left(\frac{R^2+2Rr+3r^2}{R^2+Rr+r^2}\right)$	$I^{OV} = \frac{3M}{10}\left(\frac{R^5-r^5}{R^3-r^3}\right)$
 Sphere	$A = 4\pi R^2$	$V = \frac{4\pi R^3}{3}$ $M = \frac{4\pi R^3\delta}{3}$	Center of Figure	$I_{OH} = I_{OV} = \frac{2}{5}MR^2$
 Rectangular Pyramid	Lateral Surface $A = a\sqrt{h^2+\frac{b^2}{4}}$ $+ b\sqrt{h^2+\frac{a^2}{4}}$ Base $A = ab$	$V = \frac{abh}{3}$ $M = \frac{abh\delta}{3}$	$x = \frac{h}{4}$	$I_{OG} = \frac{M}{20}\left(b^2+\frac{3}{4}h^2\right)$ $I_{OV} = \frac{M}{20}(a^2+b^2)$

PROPERTIES OF VARIOUS SOLIDS — *Continued*

SHAPE	SURFACE	VOLUME AND MASS	CENTER OF GRAVITY	MOMENT OF INERTIA
 *Sector of Sphere*	Conical Surface $A = \pi R \sqrt{2\,Rh - h^2}$	$V = \frac{2}{3}\pi R^2 h$ $M = \frac{2}{3}\pi R^2 h \delta$	$x = \frac{3}{4}\left(R - \frac{h}{2}\right)$	$I_{OV} = \frac{M}{5}(3\,Rh - h^2)$
 *Segment of Sphere*	Curved Surface $A = 2\,\pi Rh$ Base $A = \pi a^2$ $R = \frac{a^2 + h^2}{2\,h}$	$V = \pi h^2\left(R - \frac{h}{3}\right)$ $= \frac{\pi h}{6}(3\,a^2 + h^2)$ $M = \pi h^2 \delta\left(R - \frac{h}{3}\right)$	$x = \frac{3}{4}\frac{(2\,R - h)^2}{3\,R - h}$ [For surface only $x = \frac{h}{2}$]	$I_{OV} = M\left[R^2 - \frac{3\,Rh}{4} + \frac{3\,h^2}{20}\right] \times \left(\frac{2\,h}{3\,R - h}\right)$
 *Ellipsoid*	Elliptic Integral	$V = \frac{4}{3}\pi abc$ $M = \frac{4}{3}\pi abc\delta$	Center of Figure	$I_{OV} = \frac{M}{5}(b^2 + c^2)$ $I_{OH} = \frac{M}{5}(a^2 + c^2)$ $I_{OL} = \frac{M}{5}(a^2 + b^2)$

Properties of Various Solids — *Concluded*

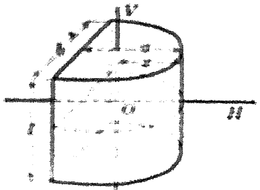
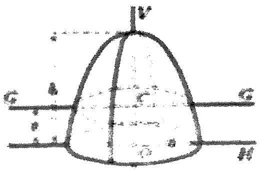
Shape	Surface	Volume and Mass	Center of Gravity	Moment of Inertia
 Parabolic Cylinder	End $A = \frac{2}{3}ab$	$V = \frac{2}{3}abl$ $M = \frac{2}{3}abl\delta$	$x = \frac{3}{5}a$	$I_{OH} = M\left[\frac{l^2}{12} + \frac{8a^2}{35}\right]$ $I_{OV} = M\left[\frac{b^2}{20} + \frac{8a^2}{35}\right]$
 Paraboloid of Revolution	Base $A = \pi r^2$	$V = \frac{\pi h r^2}{2}$ $M = \frac{\pi h r^2 \delta}{2}$	$x = \frac{h}{3}$	$I_{GG} = M\left[\frac{h^2}{18} + \frac{r^2}{[illegible]}\right]$ $I_{VV} = \frac{Mr^2}{[illegible]}$
 Circular Ring or Torus	$A = 4\pi^2 Rr$	$V = 2\pi^2 Rr^2$ $M = 2\pi^2 Rr^2\delta$	Center of figure	[illegible]

TABLE 3

Functions of Angles

Angle	Sin	Tan	Sec	Cosec	Cot	Cos	
0	0.	0.	0.	∞	∞	1.	90
1	0.0175	0.0175	1.0001	57.299	57.290	0.9998	89
2	.0349	.0349	1.0006	28.654	28.636	.9994	88
3	.0523	.0524	1.0014	19.107	19.081	.9986	87
4	.0698	.0699	1.0024	14.336	14.301	.9976	86
5	.0872	.0875	1.0038	11.474	11.430	.9962	85
6	0.1045	0.1051	1.0055	9.5668	9.5144	0.9945	84
7	.1219	.1228	1.0075	8.2055	8.1443	.9925	83
8	.1392	.1405	1.0098	7.1853	7.1154	.9903	82
9	.1564	.1584	1.0125	6.3925	6.3138	.9877	81
10	.1736	.1763	1.0154	5.7588	5.6713	.9848	80
11	0.1908	0.1944	1.0187	5.2408	5.1446	0.9816	79
12	.2079	.2126	1.0223	4.8097	4.7046	.9781	78
13	.2250	.2309	1.0263	4.4454	4.3315	.9744	77
14	.2419	.2493	1.0306	4.1336	4.0108	.9703	76
15	.2588	.2679	1.0353	3.8637	3.7321	.9659	75
16	0.2756	0.2867	1.0403	3.6280	3.4874	0.9613	74
17	.2924	.3057	1.0457	3.4203	3.2709	.9563	73
18	.3090	.3249	1.0515	3.2361	3.0777	.9511	72
19	.3256	.3443	1.0576	3.0716	2.9042	.9455	71
20	.3420	.3640	1.0642	2.9238	2.7475	.9397	70
21	0.3584	0.3839	1.0712	2.7904	2.6051	0.9336	69
22	.3746	.4040	1.0785	2.6695	2.4751	.9272	68
23	.3907	.4245	1.0864	2.5593	2.3559	.9205	67
24	.4067	.4452	1.0946	2.4586	2.2460	.9135	66
25	.4226	.4663	1.1034	2.3662	2.1445	.9063	65
26	0.4384	0.4877	1.1126	2.2812	2.0503	0.8988	64
27	.4540	.5095	1.1223	2.2027	1.9626	.8910	63
28	.4695	.5317	1.1326	2.1301	1.8807	.8829	62
29	.4848	.5543	1.1434	2.0627	1.8040	.8746	61
30	.5000	.5774	1.1547	2.0000	1.7321	.8660	60
31	0.5150	0.6009	1.1666	1.9416	1.6643	0.8572	59
32	.5299	.6249	1.1792	1.8871	1.6003	.8480	58
33	.5446	.6494	1.1924	1.8361	1.5399	.8387	57
34	.5592	.6745	1.2062	1.7883	1.4826	.8290	56
35	.5736	.7002	1.2208	1.7435	1.4281	.8192	55
36	0.5878	0.7265	1.2361	1.7013	1.3764	0.8090	54
37	.6018	.7536	1.2521	1.6616	1.3270	.7986	53
38	.6157	.7813	1.2690	1.6243	1.2799	.7880	52
39	.6293	.8098	1.2868	1.5890	1.2349	.7771	51
40	.6428	.8391	1.3054	1.5557	1.1918	.7660	50
41	0.6561	0.8693	1.3250	1.5243	1.1504	0.7547	49
42	.6691	.9004	1.3456	1.4945	1.1106	.7431	48
43	.6820	.9325	1.3673	1.4663	1.0724	.7314	47
44	.6947	.9657	1.3902	1.4396	1.0355	.7193	46
45	.7071	1.	1.4142	1.4142	1.	.7071	45
	Cos	Cot	Cosec	Sec	Tan	Sin	Angle

TABLE 4

Four-Place Logarithms of Numbers

1	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	0	1	2	3	4	5	6	7	8	9

45	0	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0	1	2	3	4	5	6	7	8	9

TABLE 5

Conversion of Logarithms

Reduction of Common Logarithms to Natural Logarithms

Rule for using Table. Divide the given common logarithm into periods of two digits and take from the table the corresponding numbers, having regard to their value as decimals. The sum will be the required natural logarithm.

Example. Find the natural logarithm corresponding to the common logarithm .497149.

Common Logarithms	Natural Logarithms
.49	1.1282667
.0071	.016348354
.000049	.00011282667
.497149	1.14472788067

Common Logarithm	Natural Logarithm	Common Logarithm	Natural Logarithm	Common Logarithm	Natural Logarithm	Common Logarithm	Natural Logarithm
1	2.30259	26	59.86721	51	117.43184	76	174.99647
2	4.60517	27	62.16980	52	119.73442	77	177.29905
3	6.90776	28	64.47238	53	122.03701	78	179.60164
4	9.21034	29	66.77497	54	124.33959	79	181.90422
5	11.51293	30	69.07755	55	126.64218	80	184.20681
6	13.81551	31	71.38014	56	128.94477	81	186.50939
7	16.11810	32	73.68272	57	131.24735	82	188.81198
8	18.42068	33	75.98531	58	133.54994	83	191.11456
9	20.72327	34	78.28789	59	135.85252	84	193.41715
10	23.02585	35	80.59048	60	138.15511	85	195.71973
11	25.32844	36	82.89306	61	140.45769	86	198.02232
12	27.63102	37	85.19565	62	142.76028	87	200.32490
13	29.93361	38	87.49823	63	145.06286	88	202.62749
14	32.23619	39	89.80082	64	147.36545	89	204.93007
15	34.53878	40	92.10340	65	149.66803	90	207.23266
16	36.84136	41	94.40599	66	151.97062	91	209.53524
17	39.14395	42	96.70857	67	154.27320	92	211.83783
18	41.44653	43	99.01116	68	156.57579	93	214.14041
19	43.74912	44	101.31374	69	158.87837	94	216.44300
20	46.05170	45	103.61633	70	161.18096	95	218.74558
21	48.35429	46	105.91891	71	163.48354	96	221.04817
22	50.65687	47	108.22150	72	165.78613	97	223.35075
23	52.95946	48	110.52408	73	168.08871	98	225.65334
24	55.26204	49	112.82667	74	170.39130	99	227.95592
25	57.56463	50	115.12925	75	172.69388	100	230.25851

TABLE 6

Conversion Tables between United States and Metric Systems

1 meter = 39.37 inches. (Act of Congress.)

Linear Measure

No.	64ths of an Inch to Millimeters	Millimeters to 64ths of an Inch	Inches to Centimeters	Centimeters to Inches
1	0.3969	2.5197	2.54	0.3937
2	0.7938	5.0394	5.08	0.7874
3	1.1906	7.5590	7.62	1.1811
4	1.5875	10.0787	10.16	1.5748
5	1.9844	12.5984	12.70	1.9685
6	2.3813	15.1181	15.24	2.3622
7	2.7781	17.6378	17.78	2.7559
8	3.1750	20.1574	20.32	3.1496
9	3.5719	22.6771	22.86	3.5433

No.	Meters to Feet	Feet to Meters	Kilometers to Miles	Miles to Kilometers
1	3.2808	0.3048	0.62137	1.60935
2	6.5617	0.6096	1.24274	3.21869
3	9.8425	0.9144	1.86411	4.82804
4	13.1233	1.2192	2.48548	6.43739
5	16.4042	1.5240	3.10685	8.04674
6	19.6850	1.8288	3.72822	9.65608
7	22.9658	2.1336	4.34959	11.26543
8	26.2467	2.4384	4.97096	12.87478
9	29.5275	2.7432	5.59233	14.48412

Areas

No.	Square Inches to Square Centimeters	Square Centimeters to Square Inches	Square Feet to Square Meters	Square Meters to Square Feet	Square Yards to Square Meters	Square Meters to Square Yards
1	6.4516	0.155	0.0929	10.7639	0.8361	1.196
2	12.9032	0.310	0.1858	21.5278	1.6722	2.392
3	19.3548	0.465	0.2787	32.2917	2.5084	3.588
4	25.8064	0.620	0.3716	43.0556	3.3445	4.784
5	32.2581	0.775	0.4645	53.8194	4.1806	5.980
6	38.7097	0.930	0.5574	64.5833	5.0167	7.176
7	45.1613	1.085	0.6503	75.3472	5.8528	8.372
8	51.6129	1.240	0.7432	86.1111	6.6890	9.568
9	58.0645	1.395	0.8361	96.8750	7.5251	10.764

No.	Acres to Hectares	Hectares to Acres	Square Miles to Square Kilometers	Square Kilometers to Square Miles	Square Miles to Hectares	Hectares to Square Miles
1	0.4047	2.471	2.59	0.3861	259.00	0.00386
2	0.8094	4.942	5.18	0.7722	518.00	0.00772
3	1.2141	7.413	7.77	1.1583	777.01	0.01158
4	1.6188	9.884	10.36	1.5444	1036.01	0.01544
5	2.0235	12.355	12.95	1.9305	1295.02	0.01930
6	2.4282	14.826	15.54	2.3166	1554.02	0.02317
7	2.8329	17.297	18.13	2.7027	1813.03	0.02703
8	3.2376	19.768	20.72	3.0887	2072.03	0.03089
9	3.6422	22.239	23.31	3.4748	2331.04	0.03475

Conversion Tables between United States and Metric Systems — *Continued*

1 kilogram = 2.2046 pounds. (Act of Congress.)

Weights

No.	Kilograms to Ounces Troy	Troy Ounces to Grams	Grains to Milligrams	Grams to Grains	Gross Tons to Metric Tons	Metric Tons to Gross Tons
1	32.1507	31.1035	64.8004	15.432	1.0161	0.9842
2	64.3015	62.2070	129.6008	30.864	2.0321	1.9684
3	96.4522	93.3104	194.4012	46.296	3.0482	2.9526
4	128.6030	124.4139	259.2017	61.728	4.0642	3.9368
5	160.7537	155.5174	324.0021	77.160	5.0803	4.9210
6	192.9045	186.6209	388.8025	92.592	6.0963	5.9052
7	225.0552	217.7244	453.6029	108.024	7.1124	6.8894
8	257.2059	248.8278	518.4033	123.456	8.1285	7.8736
9	289.3567	279.9313	583.2037	138.888	9.1445	8.8578

No.	Avoirdupois Ounces to Grams	Kilograms to Ounces Avoirdupois	Avoirdupois Pounds to Kilograms	Kilograms to Pounds Avoirdupois	Net Tons to Metric Tons	Metric Tons to Net Tons
1	28.3495	35.274	0.4536	2.2046	0.9072	1.1023
2	56.6990	70.548	0.9072	4.4092	1.8144	2.2046
3	85.0485	105.822	1.3608	6.6138	2.7215	3.3069
4	113.3980	141.096	1.8144	8.8184	3.6287	4.4092
5	141.7475	176.370	2.2680	11.0230	4.5359	5.5115
6	170.0970	211.644	2.7216	13.2276	5.4431	6.6138
7	198.4464	246.918	3.1752	15.4322	6.3503	7.7161
8	226.7959	282.192	3.6288	17.6368	7.2575	8.8184
9	255.1454	317.466	4.0824	19.8414	8.1647	9.9207

1 liter = 1.0567 quarts—Liquid Measure. 0.908 quart—Dry Measure. (Act of Congress.)

Liquid and Dry Measure

No.	Liters to Quarts, Liquid	Liters to Quarts, Dry	Quarts to Liters, Liquid	Quarts to Liters, Dry	Liters to Gallons Liquid	Gallons to Liters Liquid
1	1.0567	0.908	0.9463	1.1013	0.2642	3.7854
2	2.1134	1.816	1.8927	2.2026	0.5284	7.5707
3	3.1701	2.724	2.8390	3.3040	0.7925	11.3561
4	4.2268	3.632	3.7854	4.4053	1.0567	15.1415
5	5.2835	4.540	4.7317	5.5066	1.3209	18.9268
6	6.3402	5.448	5.6781	6.6079	1.5851	22.7122
7	7.3969	6.356	6.6244	7.7093	1.8492	26.4976
8	8.4536	7.264	7.5707	8.8106	2.1134	30.2830
9	9.5103	8.172	8.5171	9.9119	2.3776	34.0683

No.	Cubic Meters to Gallons Liquid	Gallons to Cubic Meters Liquid	Hectoliters to Bushels Dry	Bushels to Hectoliters Dry
1	264.17	0.0038	2.8375	0.3524
2	528.34	0.0076	5.6750	0.7048
3	792.51	0.0114	8.5125	1.0573
4	1056.68	0.0151	11.3500	1.4097
5	1320.85	0.0189	14.1875	1.7621
6	1585.02	0.0227	17.0250	2.1145
7	1849.19	0.0265	19.8625	2.4669
8	2113.36	0.0303	22.7000	2.8194
9	2377.53	0.0341	25.5375	3.1718

Conversion Tables between United States and Metric Systems — *Continued*

Cubic, Horsepower, and Ton Measures

No.	Cubic Centimeters to Cubic Inches	Cubic Inches to Cubic Centimeters	Cubic Meters to Cubic Feet	Cubic Feet to Cubic Meters	Cubic Meters to Cubic Yards	Cubic Yards to Cubic Meters
1	0.061	16.3934	35.316	0.0283	1.308	0.7645
2	0.122	32.7869	70.632	0.0566	2.616	1.5291
3	0.183	49.1803	105.948	0.0849	3.924	2.2936
4	0.244	65.5738	141.264	0.1133	5.232	3.0581
5	0.305	81.9672	176.580	0.1416	6.540	3.8226
6	0.366	98.3607	211.896	0.1699	7.848	4.5872
7	0.427	114.7541	247.212	0.1982	9.156	5.3517
8	0.488	131.1475	282.528	0.2265	10.464	6.1162
9	0.549	147.5410	317.844	0.2548	11.772	6.8807

No.	Horsepower Metric to U. S.	Horsepower U. S. to Metric	Foot-pounds to Kilogram-meters	Kilogram-meters to Foot-pounds	Gross Tons per Square Foot to Metric Tons per Square Meter	Metric Tons per Square Meter to Gross Tons per Square Foot
1	0.986	1.014	0.1383	7.2329	10.937	0.091
2	1.973	2.028	0.2765	14.4659	21.873	0.183
3	2.959	3.042	0.4148	21.6988	32.810	0.274
4	3.945	4.056	0.5530	28.9317	43.747	0.366
5	4.932	5.069	0.6913	36.1646	54.684	0.457
6	5.918	6.083	0.8295	43.3976	65.620	0.549
7	6.904	7.097	0.9678	50.6305	76.557	0.640
8	7.890	8.111	1.1061	57.8634	87.494	0.731
9	8.877	9.125	1.2443	65.0963	98.431	0.823

Miscellaneous

No.	Kilograms per Meter to Pounds per Foot	Pounds per Foot to Kilograms per Meter	Kilograms per Square Meter to Pounds per Square Foot	Pounds per Square Foot to Kilograms per Square Meter
1	0.6720	1.4882	0.2048	4.8825
2	1.3439	2.9764	0.4096	9.7649
3	2.0159	4.4645	0.6144	14.6474
4	2.6879	5.9527	0.8193	19.5299
5	3.3598	7.4409	1.0241	24.4123
6	4.0318	8.9291	1.2289	29.2948
7	4.7037	10.4172	1.4337	34.1773
8	5.3757	11.9054	1.6385	39.0597
9	6.0477	13.3936	1.8433	43.9422

No.	Kilograms per Cubic Meter to Pounds per Cubic Foot	Pounds per Cubic Foot to Kilograms per Cubic Meter	Kilograms per Square Centimeters to Pounds per Square Inch	Pounds per Square Inch to Kilograms per Square Centimeter
1	0.0624	16.0192	14.2232	0.0703
2	0.1248	32.0385	28.4465	0.1406
3	0.1873	48.0577	42.6697	0.2109
4	0.2497	64.0769	56.8929	0.2812
5	0.3121	80.0962	71.1161	0.3515
6	0.3745	96.1154	85.3394	0.4218
7	0.4370	112.1346	99.5626	0.4922
8	0.4994	128.1539	113.7858	0.5625
9	0.5618	144.1731	128.0090	0.6328

TABLE 7

Weights and Measures

Avoirdupois or Ordinary Commercial Weight

United States and British

Gross Tons	Cwts	Pounds	Ounces
1.	20.	2240.	35840
0.050	1.	112.	1792
	0.0089	1	16
		0.0625	1

1 pound = 27.7 cubic inches of distilled water at its maximum density (39° Fahrenheit).

Linear Measure

United States and British

Miles	Rods	Yards	Feet	Inches
1.	320.	1760.	5280	63,360
0.003125	1.	5.5	16.5	198
0.000568	0.1818	1	3	36
0.0001894	0.0606	0.3333	1	12
0.0000158	0.005051	0.02778	0.08333	1

The British measures are shorter than those of the United States by about 1 part in 17,230 or 3.677 inches in a mile.

A fathom = 6 feet. A Gunter's surveying chain = 66 feet or 4 rods, 80 chains making a mile.

Area, or Land, Measure

United States and British

Square Miles	Acres	Square Rods	Square Yards	Square Feet	Square Inches
1.	640.	102400	3097600	27878400	
	1.	160.	4840.	43560	6272640
		1	30.25	272.25	39204
		0.0331	1	9.0	1296
			0.111	1	144
				[illegible]	[illegible]

WEIGHTS AND MEASURES — *Continued*

CUBIC, OR SOLID, MEASURE

United States and British

1728 cubic inches = 1 cubic foot.
27 cubic feet = 1 cubic yard.
A cord of wood = 4′ × 4′ × 8′ = 128 cubic feet.
A perch of masonry = 16.5′ × 1.5′ × 1′ = 24.75 cubic feet, but is generally assumed at 25 cubic feet.

DRY MEASURE

United States Only

Struck Bush.	Pecks	Quarts	Pints	Gallons	Cubic Inches
1	4	32.	64	8.	2150.
	1	8.	16	2.	537.6
		1.	2	0.25	67.2
		0.5	1	0.125	33.6
		4.	8	1.	268.8

A gallon of liquid measure = 231 cubic inches.
A heaped bushel = $1\frac{1}{4}$ struck bushels. The cone in a heaped bushel must be not less than 6 inches high.
A barrel of U. S. hydraulic cement = 300 to 310 pounds usually and of genuine Portment cement = 425 pounds.
To reduce U. S. dry measures to British imperial of the same name divide by 1.032.

NAUTICAL MEASURE

A nautical or sea mile is the length of a minute of longitude of the earth at the equator at the level of the sea. It is assumed as 6080.07 feet = 1.152664 statute or land miles by the U. S. Coast Survey.
3 nautical miles = 1 league.
1 knot = 1 nautical mile per hour.

TABLE 8

Fractions of a Lineal Inch in Decimals

Fractions	Decimals of an Inch	Fractions	Decimals of an Inch	Fractions	Decimals of an Inch	Fractions	Decimals of an Inch
1/64	0.015625	17/64	0.265625	33/64	0.515625	49/64	0.765625
1/32	0.03125	9/32	0.28125	17/32	0.53125	25/32	0.78125
3/64	0.04687	19/64	0.296875	35/64	0.546875	51/64	0.796875
1/16	0.0625	5/16	0.3125	9/16	0.5625	13/16	0.8125
5/64	0.078125	21/64	0.328125	37/64	0.578125	53/64	0.828125
3/32	0.09375	11/32	0.34375	19/32	0.59375	27/32	0.84375
7/64	0.109375	23/64	0.359375	39/64	0.609375	55/64	0.859375
1/8	0.125	3/8	0.375	5/8	0.625	7/8	0.875
9/64	0.140625	25/64	0.390625	41/64	0.640625	57/64	0.890625
5/32	0.15625	13/32	0.40625	21/32	0.65625	29/32	0.90625
11/64	0.171875	27/64	0.421875	43/64	0.671875	59/64	0.921875
3/16	0.1875	7/16	0.4375	11/16	0.6875	15/16	0.9375
13/64	0.203125	29/64	0.453125	45/64	0.703125	61/64	0.953125
7/32	0.21875	15/32	0.46875	23/32	0.71875	31/32	0.96875
15/64	0.234375	31/64	0.484375	47/64	0.734375	63/64	0.984375
1/4	0.25	1/2	0.5	3/4	0.75	1	1.000

Lineal Inches in Decimal Fractions of a Lineal Foot

Lineal Inches	Lineal Foot	Lineal Inches	Lineal Foot	Lineal Inches	Lineal Foot
1/64	0.001302083	1 7/8	0.15625	6 3/4	0.5625
1/32	0.00260416	2	0.1666	7	0.5833
1/16	0.0052083	2 1/8	0.177083	7 1/4	0.60416
1/8	0.010416	2 1/4	0.1875	7 1/2	0.625
3/16	0.015625	2 3/8	0.197916	7 3/4	0.64583
1/4	0.02083	2 1/2	0.2083	8	0.66667
5/16	0.0260416	2 5/8	0.21875	8 1/4	0.6875
3/8	0.03125	2 3/4	0.22916	8 1/2	0.7083
7/16	0.0364583	2 7/8	0.239583	8 3/4	0.72916
1/2	0.0416	3	0.25	9	0.75
9/16	0.046875	3 1/4	0.27083	9 1/4	0.77083
5/8	0.052083	3 1/2	0.2916	9 1/2	0.7916
11/16	0.0572916	3 3/4	0.3125	9 3/4	0.8125
3/4	0.0625	4	0.33333	10	0.83333
13/16	0.0677083	4 1/4	0.35416	10 1/4	0.85416
7/8	0.072916	4 1/2	0.375	10 1/2	0.875
15/16	0.078125	4 3/4	0.39583	10 3/4	0.89583
1	0.0833	5	0.4166	11	0.9166
1 1/8	0.09375	5 1/4	0.4375	11 1/4	0.9375
1 1/4	0.10416	5 1/2	0.4583	11 1/2	0.9583
1 3/8	0.114583	5 3/4	0.47916	11 3/4	0.97916
1 1/2	0.125	6	0.5	12	1.000
1 5/8	0.135416	6 1/4	0.52083	....	....
1 3/4	0.14583	6 1/2	0.5416	....	....

TABLE 9

Circumferences and Areas of Circles

Diameters, $\frac{1}{16}$ inch up to and including 120 inches. Advancing, $\frac{1}{16}$ to 1; $\frac{1}{8}$ to 50; $\frac{1}{4}$ to 80; and $\frac{1}{2}$ to 120.

Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches
$\frac{1}{16}$	0.19635	0.00307	$4\frac{1}{2}$	14.137	15.904	$9\frac{5}{8}$	30.237	72.759
$\frac{1}{8}$	0.3927	0.01227	$4\frac{5}{8}$	14.529	16.800	$9\frac{3}{4}$	30.630	74.662
$\frac{3}{16}$	0.5890	0.02761	$4\frac{3}{4}$	14.922	17.720	$9\frac{7}{8}$	31.023	76.588
$\frac{1}{4}$	0.7854	0.04909	$4\frac{7}{8}$	15.315	18.665	10	31.416	78.540
$\frac{5}{16}$	0.9817	0.07670	5	15.708	19.635	$10\frac{1}{8}$	31.808	80.515
$\frac{3}{8}$	1.1781	0.1104	$5\frac{1}{8}$	16.100	20.629	$10\frac{1}{4}$	32.201	82.516
$\frac{7}{16}$	1.3744	0.1503	$5\frac{1}{4}$	16.493	21.647	$10\frac{3}{8}$	32.594	84.540
$\frac{1}{2}$	1.5708	0.1963	$5\frac{3}{8}$	16.886	22.690	$10\frac{1}{2}$	32.986	86.590
$\frac{9}{16}$	1.7771	0.2485	$5\frac{1}{2}$	17.278	23.758	$10\frac{5}{8}$	33.379	88.664
$\frac{5}{8}$	1.9635	0.3068	$5\frac{5}{8}$	17.671	24.850	$10\frac{3}{4}$	33.772	90.762
$\frac{11}{16}$	2.1598	0.3712	$5\frac{3}{4}$	18.064	25.967	$10\frac{7}{8}$	34.164	92.885
$\frac{3}{4}$	2.3562	0.4417	$5\frac{7}{8}$	18.457	27.108	11	34.558	95.033
$\frac{13}{16}$	2.5525	0.5185	6	18.849	28.274	$11\frac{1}{8}$	34.950	97.205
$\frac{7}{8}$	2.7489	0.6013	$6\frac{1}{8}$	19.242	29.464	$11\frac{1}{4}$	35.343	99.402
$\frac{15}{16}$	2.9452	0.6903	$6\frac{1}{4}$	19.635	30.679	$11\frac{3}{8}$	35.735	101.623
1	3.1416	0.7854	$6\frac{3}{8}$	20.027	31.919	$11\frac{1}{2}$	36.128	103.869
$1\frac{1}{8}$	3.5343	0.9940	$6\frac{1}{2}$	20.420	33.183	$11\frac{5}{8}$	36.521	106.139
$1\frac{1}{4}$	3.9270	1.2271	$6\frac{5}{8}$	20.813	34.471	$11\frac{3}{4}$	36.913	108.434
$1\frac{3}{8}$	4.3197	1.4848	$6\frac{3}{4}$	21.205	35.784	$11\frac{7}{8}$	37.306	110.753
$1\frac{1}{2}$	4.1724	1.7671	$6\frac{7}{8}$	21.598	37.122	12	37.699	113.097
$1\frac{5}{8}$	5.1051	2.0739	7	21.991	38.484	$12\frac{1}{8}$	38.091	115.466
$1\frac{3}{4}$	5.4978	2.4052	$7\frac{1}{8}$	22.383	39.871	$12\frac{1}{4}$	38.484	117.859
$1\frac{7}{8}$	5.8905	2.7621	$7\frac{1}{4}$	22.776	41.282	$12\frac{3}{8}$	38.877	120.276
2	6.2832	3.1416	$7\frac{3}{8}$	23.169	42.718	$12\frac{1}{2}$	39.270	122.718
$2\frac{1}{8}$	6.6759	3.5465	$7\frac{1}{2}$	23.562	44.178	$12\frac{5}{8}$	39.662	125.184
$2\frac{1}{4}$	7.0686	3.9760	$7\frac{5}{8}$	23.954	45.663	$12\frac{3}{4}$	40.055	127.676
$2\frac{3}{8}$	7.4613	4.4302	$7\frac{3}{4}$	24.347	47.173	$12\frac{7}{8}$	40.448	130.192
$2\frac{1}{2}$	7.8540	4.9087	$7\frac{7}{8}$	24.740	48.707	13	40.840	132.732
$2\frac{5}{8}$	8.2467	5.4119	8	25.132	50.265	$13\frac{1}{8}$	41.233	135.297
$2\frac{3}{4}$	8.6394	5.9395	$8\frac{1}{8}$	25.515	51.848	$13\frac{1}{4}$	41.626	137.886
$2\frac{7}{8}$	9.0321	6.4918	$8\frac{1}{4}$	25.918	53.456	$13\frac{3}{8}$	42.018	140.500
3	9.4248	7.0686	$8\frac{3}{8}$	26.310	55.088	$13\frac{1}{2}$	42.411	143.139
$3\frac{1}{8}$	9.8175	7.6699	$8\frac{1}{2}$	26.703	56.745	$13\frac{5}{8}$	42.804	145.802
$3\frac{1}{4}$	10.210	8.2957	$8\frac{5}{8}$	27.096	58.426	$13\frac{3}{4}$	43.197	148.489
$3\frac{3}{8}$	10.602	8.9462	$8\frac{3}{4}$	27.489	60.132	$13\frac{7}{8}$	43.590	151.201
$3\frac{1}{2}$	10.995	9.6211	$8\frac{7}{8}$	27.881	61.862	14	43.982	153.938
$3\frac{5}{8}$	11.388	10.320	9	28.274	63.617	$14\frac{1}{8}$	44.375	156.700
$3\frac{3}{4}$	11.781	11.044	$9\frac{1}{8}$	28.667	65.397	$14\frac{1}{4}$	44.767	159.485
$3\frac{7}{8}$	12.173	11.793	$9\frac{1}{4}$	29.059	67.201	$14\frac{3}{8}$	45.160	162.295
4	12.566	12.566	$9\frac{3}{8}$	29.452	69.029	$14\frac{1}{2}$	45.553	165.130
$4\frac{1}{8}$	12.959	13.364	$9\frac{1}{2}$	29.845	70.882	$14\frac{5}{8}$	45.945	167.989
$4\frac{1}{4}$	13.351	14.186				$14\frac{3}{4}$	46.338	170.873
$4\frac{3}{8}$	13.744	15.033				$14\frac{7}{8}$	46.731	173.782

Circumferences and Areas of Circles (Continued)

Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches
15	47.124	176.715	21	65.973	346.361	27	84.823	572.555
15⅛	47.516	179.672	21⅛	66.366	350.497	27⅛	85.215	577.870
15¼	47.909	182.654	21¼	66.759	354.657	27¼	85.608	583.208
15⅜	48.302	185.661	21⅜	67.151	358.841	27⅜	86.001	588.571
15½	48.694	188.692	21½	67.544	363.051	27½	86.394	593.957
15⅝	49.087	191.748	21⅝	67.937	367.284	27⅝	86.786	599.369
15¾	49.480	194.828	21¾	68.329	371.543	27¾	87.179	604.806
15⅞	49.872	197.933	21⅞	68.722	375.826	27⅞	87.572	610.267
16	50.265	201.062	22	69.115	380.133	28	87.965	615.752
16⅛	50.658	204.216	22⅛	69.507	384.465	28⅛	88.357	621.263
16¼	51.051	207.394	22¼	69.900	388.822	28¼	88.750	626.797
16⅜	51.443	210.597	22⅜	70.293	393.203	28⅜	89.143	632.356
16½	51.836	213.825	22½	70.686	397.608	28½	89.535	637.940
16⅝	52.229	217.077	22⅝	71.078	402.038	28⅝	89.928	643.548
16¾	52.621	220.353	22¾	71.471	406.493	28¾	90.321	649.181
16⅞	53.014	223.654	22⅞	71.864	410.972	28⅞	90.713	654.838
17	53.407	226.980	23	72.256	415.476	29	91.106	660.520
17⅛	53.799	230.330	23⅛	72.649	420.004	29⅛	91.499	666.226
17¼	54.192	233.705	23¼	73.042	424.557	29¼	91.892	671.957
17⅜	54.585	237.104	23⅜	73.434	429.135	29⅜	92.284	677.713
17½	54.978	240.528	23½	73.827	433.731	29½	92.677	683.493
17⅝	55.370	243.977	23⅝	74.220	438.363	29⅝	93.070	689.297
17¾	55.763	247.450	23¾	74.613	443.014	29¾	93.462	695.126
17⅞	56.156	250.947	23⅞	75.005	447.690	29⅞	93.855	700.980
18	56.548	254.469	24	75.398	452.390	30	94.248	706.858
18⅛	56.941	258.016	24⅛	75.791	457.115	30⅛	94.640	712.761
18¼	57.334	261.586	24¼	76.183	461.864	30¼	95.033	718.688
18⅜	57.726	265.182	24⅜	76.576	466.638	30⅜	95.426	724.640
18½	58.119	268.803	24½	76.969	471.436	30½	95.819	730.617
18⅝	58.512	272.447	24⅝	77.361	476.259	30⅝	96.211	736.618
18¾	58.905	276.117	24¾	77.754	481.106	30¾	96.604	742.643
18⅞	59.297	279.811	24⅞	78.147	485.978	30⅞	96.997	748.693
19	59.690	283.529	25	78.540	490.875	31	97.389	754.768
19⅛	60.083	287.272	25⅛	78.932	495.796	31⅛	97.782	760.867
19¼	60.475	291.039	25¼	79.325	500.741	31¼	98.175	766.990
19⅜	60.868	294.831	25⅜	79.718	505.711	31⅜	98.567	773.139
19½	61.261	298.648	25½	80.110	510.706	31½	98.960	779.311
19⅝	61.653	302.489	25⅝	80.503	515.725	31⅝	99.353	785.509
19¾	62.046	306.355	25¾	80.896	520.769	31¾	99.746	791.730
19⅞	62.439	310.245	25⅞	81.288	525.837	31⅞	100.138	797.977
20	62.832	314.160	26	81.681	530.930	32	100.531	804.248
20⅛	63.224	318.099	26⅛	82.074	536.047	32⅛	100.924	810.543
20¼	63.617	322.063	26¼	82.467	541.189	32¼	101.316	816.863
20⅜	64.010	326.051	26⅜	82.859	546.356	32⅜	101.709	823.208
20½	64.402	330.064	26½	83.252	551.547	32½	102.102	829.577
20⅝	64.795	334.101	26⅝	83.645	556.762	32⅝	102.494	835.970
20¾	65.188	338.163	26¾	84.037	562.002	32¾	102.887	842.389
20⅞	65.580	342.250	26⅞	84.430	567.267	32⅞	103.280	848.831

CIRCUMFERENCES AND AREAS OF CIRCLES—*Continued*

DIAMETER INCHES	CIRCUMFERENCE INCHES	AREA SQUARE INCHES	DIAMETER INCHES	CIRCUMFERENCE INCHES	AREA SQUARE INCHES	DIAMETER INCHES	CIRCUMFERENCE INCHES	AREA SQUARE INCHES
33	103.672	855.30	39	122.522	1194.59	45	141.372	1590.43
33⅛	104.065	861.79	39⅛	122.915	1202.26	45⅛	141.764	1599.28
33¼	104.458	868.30	39¼	123.307	1209.95	45¼	142.157	1608.15
33⅜	104.850	874.84	39⅜	123.700	1217.67	45⅜	142.550	1617.04
33½	105.243	881.41	39½	124.093	1225.42	45½	142.942	1625.97
33⅝	105.636	888.00	39⅝	124.485	1233.18	45⅝	143.335	1634.92
33¾	106.029	894.61	39¾	124.878	1240.98	45¾	143.728	1643.89
33⅞	106.421	901.25	39⅞	125.271	1248.79	45⅞	144.120	1652.88
34	106.814	907.92	40	125.664	1256.64	46	144.513	1661.90
34⅛	107.207	914.61	40⅛	126.056	1264.50	46⅛	144.906	1670.95
34¼	107.599	921.32	40¼	126.449	1272.39	46¼	145.299	1680.01
34⅜	107.992	928.06	40⅜	126.842	1280.31	46⅜	145.691	1689.10
34½	108.385	934.82	40½	127.234	1288.25	46½	146.084	1698.23
34⅝	108.777	941.60	40⅝	127.627	1296.21	46⅝	146.477	1707.37
34¾	109.170	948.41	40¾	128.020	1304.20	46¾	146.869	1716.54
34⅞	109.563	955.25	40⅞	128.412	1312.21	46⅞	147.262	1725.73
35	109.956	962.11	41	128.805	1320.25	47	147.655	1734.94
35⅛	110.348	968.99	41⅛	129.198	1328.32	47⅛	148.047	1744.18
35¼	110.741	975.90	41¼	129.591	1336.40	47¼	148.440	1753.45
35⅜	111.134	982.84	41⅜	129.983	1344.51	47⅜	148.833	1762.73
35½	111.526	989.80	41½	130.376	1352.65	47½	149.226	1772.05
35⅝	111.919	996.78	41⅝	130.769	1360.81	47⅝	149.618	1781.39
35¾	112.312	1003.78	41¾	131.161	1369.00	47¾	150.011	1790.76
35⅞	112.704	1010.82	41⅞	131.554	1377.21	47⅞	150.404	1800.14
36	113.097	1017.88	42	131.947	1385.44	48	150.796	1809.56
36⅛	113.490	1024.95	42⅛	132.339	1393.70	48⅛	151.189	1818.99
36¼	113.883	1032.06	42¼	132.732	1401.98	48¼	151.582	1828.46
36⅜	114.275	1039.19	42⅜	133.125	1410.29	48⅜	151.974	1837.93
36½	114.668	1046.35	42½	133.518	1418.62	48½	152.367	1847.45
36⅝	115.061	1053.52	42⅝	133.910	1426.98	48⅝	152.760	1856.99
36¾	115.453	1060.73	42¾	134.303	1435.36	48¾	153.153	1866.55
36⅞	115.846	1067.95	42⅞	134.696	1443.77	48⅞	153.545	1876.13
37	116.239	1075.21	43	135.088	1452.20	49	153.938	1885.74
37⅛	116.631	1082.48	43⅛	135.481	1460.65	49⅛	154.331	1895.37
37¼	117.024	1089.79	43¼	135.874	1469.13	49¼	154.723	1905.03
37⅜	117.417	1097.11	43⅜	136.266	1477.63	49⅜	155.116	1914.70
37½	117.810	1104.46	43½	136.659	1486.17	49½	155.509	1924.42
37⅝	118.202	1111.84	43⅝	137.052	1494.72	49⅝	155.901	1934.15
37¾	118.595	1119.24	43¾	137.445	1503.30	49¾	156.294	1943.91
37⅞	118.988	1126.66	43⅞	137.837	1511.90	49⅞	156.687	1953.69
38	119.380	1134.11	44	138.230	1520.53	50	157.080	1963.50
38⅛	119.773	1141.59	44⅛	138.623	1529.18	50¼	157.865	1983.18
38¼	120.166	1149.08	44¼	139.015	1537.86	50½	158.650	2002.96
38⅜	120.558	1156.61	44⅜	139.408	1546.55	50¾	159.436	2022.84
38½	120.951	1164.15	44½	139.801	1555.28	51	160.221	2042.82
38⅝	121.344	1171.73	44⅝	140.193	1564.03	51¼	161.007	2062.90
38¾	121.737	1179.32	44¾	140.586	1572.81	51½	161.792	2083.07
38⅞	122.129	1186.94	44⅞	140.979	1581.61	51¾	162.577	2103.35

Circumferences and Areas of Circles — *Continued*

Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches
52	163.363	2123.72	63	197.920	3117.25	74	232.478	4300.84
52¼	164.148	2144.19	63¼	198.706	3142.04	74¼	233.263	4329.85
52½	164.934	2164.75	63½	199.491	3166.92	74½	234.049	4359.16
52¾	165.719	2185.42	63¾	200.277	3191.91	74¾	234.834	4388.47
53	166.504	2206.18	64	201.062	3216.99	75	235.620	4417.86
53¼	167.290	2227.05	64¼	201.847	3242.17	75¼	236.405	4447.37
53½	168.075	2248.01	64½	202.633	3267.46	75½	237.190	4476.97
53¾	168.861	2269.06	64¾	203.418	3292.83	75¾	237.976	4506.67
54	169.646	2290.22	65	204.204	3318.31	76	238.761	4536.46
54¼	170.431	2311.48	65¼	204.989	3343.88	76¼	239.547	4566.36
54½	171.217	2332.83	65½	205.774	3369.56	76½	240.332	4596.35
54¾	172.002	2354.28	65¾	206.560	3395.33	76¾	241.117	4626.44
55	172.788	2375.83	66	207.345	3421.19	77	241.903	4656.63
55¼	173.573	2397.48	66¼	208.131	3447.16	77¼	242.688	4686.92
55½	174.358	2419.22	66½	208.916	3473.23	77½	243.474	4717.30
55¾	175.144	2441.07	66¾	209.701	3499.39	77¾	244.259	4747.79
56	175.929	2463.01	67	210.487	3525.65	78	245.044	4778.36
56¼	176.715	2485.05	67¼	211.272	3552.01	78¼	245.830	4809.05
56½	177.500	2507.19	67½	212.058	3578.47	78½	246.615	4839.83
56¾	178.285	2529.42	67¾	212.843	3605.03	78¾	247.401	4870.70
57	179.071	2551.76	68	213.628	3631.68	79	248.186	4901.68
57¼	179.856	2574.19	68¼	214.414	3658.44	79¼	248.971	4932.75
57½	180.642	2596.72	68½	215.199	3685.29	79½	249.757	4963.92
57¾	181.427	2619.35	68¾	215.985	3712.24	79¾	250.542	4995.19
58	182.212	2642.08	69	216.770	3739.28	80	251.328	5026.55
58¼	182.998	2664.91	69¼	217.555	3766.43	80½	252.898	5089.58
58½	183.783	2687.83	69½	218.341	3793.67	81	254.469	5153.00
58¾	184.569	2710.85	69¾	219.126	3821.02	81½	256.040	5216.82
59	185.354	2733.97	70	219.912	3848.45	82	257.611	5281.02
59¼	186.139	2757.19	70¼	220.697	3875.99	82½	259.182	5345.62
59½	186.925	2780.51	70½	221.482	3903.63	83	260.752	5410.61
59¾	187.710	2803.92	70¾	222.268	3931.36	83½	262.323	5476.00
60	188.496	2827.43	71	223.053	3959.19	84	263.894	5541.77
60¼	189.281	2851.05	71¼	223.839	3987.13	84½	265.465	5607.95
60½	190.066	2874.76	71½	224.624	4015.15	85	267.035	5674.51
60¾	190.852	2898.56	71¾	225.409	4043.28	85½	268.606	5741.47
61	191.637	2922.47	72	226.195	4071.50	86	270.177	5808.80
61¼	192.423	2946.47	72¼	226.980	4099.83	86½	271.748	5876.55
61½	193.208	2970.57	72½	227.766	4128.25	87	273.319	5944.68
61¾	193.993	2994.77	72¾	228.551	4156.77	87½	274.889	6013.21
62	194.779	3019.07	73	229.336	4185.39	88	276.460	6082.12
62¼	195.564	3043.47	73¼	230.122	4214.11	88½	278.031	6151.44
62½	196.350	3067.96	73½	230.907	4242.92			
62¾	197.135	3092.56	73¾	231.693	4271.83			

CIRCUMFERENCES AND AREAS OF CIRCLES — *Continued*

Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches	Diameter Inches	Circumference Inches	Area Square Inches
89	279.602	6221.14	100	314.159	7853.98	111	348.717	9676.89
89½	281.173	6291.25	100½	315.730	7932.72	111½	350.288	9764.28
90	282.744	6361.73	101	317.301	8011.85	112	351.858	9852.03
90½	284.314	6432.62	101½	318.872	8091.36	112½	353.430	9940.20
91	285.885	6503.88	102	320.442	8171.28	113	355.000	10028.75
91½	287.456	6575.56	102½	322.014	8251.60	113½	356.570	10117.68
92	289.027	6647.61	103	323.584	8332.29	114	358.142	10207.03
92½	290.598	6720.07	103½	325.154	8413.40	142½	359.712	10296.76
93	292.168	6792.91	104	326.726	8494.87	115	361.283	10386.89
93½	293.739	6866.16	104½	328.296	8576.76	115½	362.854	10477.40
94	295.310	6939.78	105	329.867	8659.01	116	364.425	10568.32
94½	296.881	7013.81	105½	331.438	8741.68	116½	365.996	10659.64
95	298.452	7088.22	106	333.009	8824.73	117	367.566	10751.32
95½	300.022	7163.04	106½	334.580	8908.20	117½	369.138	10843.40
96	301.593	7238.23	107	336.150	8992.02	118	370.708	10935.88
96½	302.164	7313.84	107½	337.722	9076.24	118½	372.278	11028.76
97	304.734	7389.81	108	339.292	9160.88	119	373.849	11122.02
97½	306.306	7474.20	108½	340.862	9245.92	119½	375.420	11215.68
98	307.876	7542.96	109	342.434	9331.32	120	376.991	11309.73
98½	309.446	7620.12	109½	344.004	9417.12	....	......	......
99	311.018	7697.69	110	345.575	9503.32	....	......	......
99½	312.588	7775.64	110½	347.146	9589.92	....	......	......

TABLE 10

Weights of Various Substances

	Material	Weight in Lb per Ft³
Brick and Brickwork	Pressed brick	150
	Common hard brick	125
	Soft inferior brick	100
	Good pressed brick masonry	140
	Ordinary brickwork	125
Stone and Masonry	Gneiss, solid	168
	Gneiss, loose piles	96
	Granite	170
	Limestone and marble	165
	Limestone and marble, loose, broken	96
	Sandstone, solid	151
	Sandstone, quarried and piled	86
	Shale	162
	Slate	175
	Granite or limestone masonry, well dressed	165
	Granite or limestone masonry, mortar rubble	154
	Granite or limestone masonry, well scabbled dry rubble	138
	Sandstone masonry, well dressed	144
Earth, Sand, and Gravel	Common loam, dry, loose	76
	Common loam, moderately rammed	95
	Soft flowing mud	110
	Dry hard mud	90–110
	Gravel or sand, dry, loose	90–106
	Gravel or sand, well shaken	99–117
	Gravel or sand, wet	120–140
Metals	Aluminium	162
	Brass, cast (copper and zinc)	504
	Brass, rolled	524
	Bronze (copper 8, tin 1)	529
	Copper, cast	542
	Copper, rolled	555
	Iron, cast	450
	Iron, wrought	480
	Lead	710
	Platinum	1342
	Steel	489.6
	Tin, cast	459
	Zinc, spelter	437.5
Coal	Anthracite, solid Penn.	93
	Anthracite, broken, loose (heaped bushel 80)	54
	Anthracite, broken, shaken	58
	Bituminous, solid	84
	Bituminous, loose (heaped bushel 74)	49
	Bituminous, broken, shaken	51–56
	Coke, loose (heaped bushel 40)	26

Weights of Various Substances — *Continued*

	Material	Weight in Lb. Ft.^{3}
Lime d Hydraulic Cement	Quicklime, loose (struck bushel 66)	53
	Quicklime, well shaken	75
	American Louisville, loose	50
	American Rosendale, loose	56
	American Cumberland, loose	65
	American Cumberland, well shaken	85
	English Portland	90
	American Portland, loose	88
	American Portland, well shaken	110
Dry Wood	Ash, American White	38
	Cherry	42
	Chestnut	41
	Cypress	64
	Ebony	76
	Elm	35
	Hemlock	25
	Hickory	53
	Lignum-vitæ	83
	Locust	44
	Mahogany, Spanish	53
	Mahogany, Honduras	35
	Maple	49
	Oak, Live	59
	Oak, Red or Black	32–45
	Oak, White	48
	Pine, White	25
	Pine, Yellow Northern	34
	Pine, Yellow Southern	45
	Poplar	29
	Sycamore	37
	Spruce	25
	Walnut, Black	38

TABLE 11

Heat

Substance	Heat of Fusion B. T. U. per lb.	Specific Heat	Melting Point in Degrees Fahrenheit
Aluminum		0.212	1100
Bismuth	22.7	0.0305	500
Cadmium	24.5	0.055	600
Copper	....	0.093	1930
Gold	....	0.032	1910
Ice	144	0.504	32
Iridium	....	.....	3540
Iron, pig	....	0.116 0.119	1970
Iron, pure	....	0.113	2800
Iron, wrought	....	0.108	2500
Lead	10.4	0.0315	620
Mercury	5.08	0.033	40
Nickel	8.3	0.109	2640
Platinum	49.0	0.032	3230
Silver	44.5	0.056	1750
Tin	26.3	0.054	446
Zinc	50.6	0.0935	774

Heat Units

Amount of heat required to raise the temperature of 1 gram of water from 0° to 1° Centigrade is called the **calorie.**

1 British Thermal Unit (B. T. U.) = 252.1 calories.

1 Watt = 1 Joule/sec. = 0.239 calorie/sec.

Latent heat of vaporization of water (100° C.) = 537 calories per kilogram.

Latent heat of vaporization of water (212° F.) = 966.6 B. T. U. per pound.

1 H. P. per hour $= \frac{33000 \times 60}{778} = 2545$ B. T. U.

Mechanical Equivalent of Heat

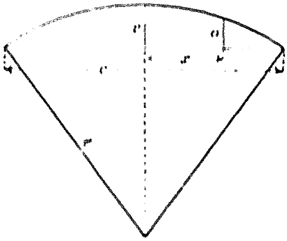
Energy required to raise the temperature of 1 gram of water from 20° to 21° C. $= 4.18 \times 10^7$ ergs. Energy required to raise the temperature of 1 lb. of water from 68° to 69° F. = 778 ft.-lb. This is known as **Joule's equivalent**.

TABLE 12

Mensuration

Circular Measure

Circumference of circle = diameter × 3.1416.
Diameter of circle = circumference × 0.3183.
Side of square of same periphery as circle = diameter × 0.7854.
Diameter of circle of same periphery as square = side × 1.2732.
Side of an inscribed square = diameter of circle × 0.7071.
Length of arc = number of degrees × diameter × 0.008727.
Circumference of circle whose diameter is 1 = π.



$$r = \frac{v^2 + \frac{c^2}{4}}{2v},$$

very nearly $= \frac{c^2}{8v}$.

r = radius,
c = chord,
v = versine,
o = ordinate.

$$o = \sqrt{r^2 - x^2} - (r - v).$$

$$v = r - \sqrt{r^2 - \frac{c^2}{4}}, \text{ or very nearly} = \frac{c^2}{8r}.$$

Number	Common Logarithm
π = 3.14159265	0.4971499
$\sqrt{\pi}$ = 1.772454	0.24857494
π^2 = 9.869604	0.99429995
$\frac{1}{\pi}$ = 0.318310	9.50285013 − 10
$\frac{1}{\pi^2}$ = 0.101321	9.00570025 − 10
$\frac{1}{\sqrt{\pi}}$ = 0.564190	9.75142506 − 10

1 radian = angle subtended by circular arc equal in length to the radius of the circle;

$$\pi \text{ radians} = 180 \text{ degrees};$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ = 57.29577951^\circ.$$

Segments of circle = area of sector less triangle; also for flat segments very nearly $= \frac{4v}{3}\sqrt{0.388v^2 + \frac{c^2}{4}}$.

Side of square of approximately same area as circle = diameter × 0.8862; also circumference × 0.2821.

Diameter of circle of same area as square = side × 1.1284 approximately.

Area of parabola = base × ⅔ height.

Area of ellipse = long diameter × short diameter × 0.7854.

Area of regular polygon = sum of sides × half perpendicular distance from center to sides.

MENSURATION — *Continued*

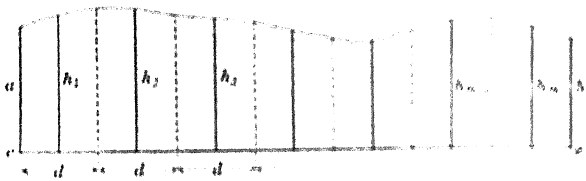
Surface of cylinder = circumference × height + area of both ends.

Surface of sphere = diameter squared × 3.1416; also = circumference × diameter.

Surface of a right pyramid or cone = periphery or circumference of base × half slant height.

Surface of a frustum of a regular right pyramid or cone = sum of peripheries or circumferences of the two ends × half slant height + area of both ends.

The following formulæ are used to obtain the areas of irregular plane surfaces which are bounded by a base line *ce* and two ordinates *a* and *b* as per figure.



The formulæ are given in the order of their accuracy, beginning with the most accurate.

The surface is divided into any number (n) of parallel strips having the same widths d and whose middle ordinates are represented by $h_1, h_2, h_3, \ldots h_{n-1}$ and h_n.

I. Area $= d \times \Sigma h + \frac{d}{72}(8a + h_2 - 9h_1) + \frac{d}{72}(8b + h_{n-1} - 9h_n)$ (Francke's rule.)

II. Area $= d \times \Sigma h + \frac{d}{12}(a - h_1) + \frac{d}{12}(b - h_n)$ (Poncelet's rule.)

III. Area $= d \times \Sigma h$.

SOLID CONTENTS

Prism, right or oblique, = area of base × perpendicular height.

Cylinder, right or oblique, = area of section at right angles to sides × length of side.

Sphere = diameter cubed × 0.5236; also = surface × ⅙ diameter.

Pyramid or cone, right or oblique, regular or irregular, = area of base × ⅓ perpendicular height.

PRISMOIDAL FORMULA

A prismoid is a solid bounded by six plane surfaces only two of which are parallel.

To find the contents of a prismoid, add together the areas of the two parallel surfaces and four times the area of a section taken midway between and parallel to them, and multiply the sum by ⅙ of the perpendicular distance between the parallel surfaces.

AREA

Triangle = base × half perpendicular height.

Parallelogram = base × perpendicular height.

Trapezoid = half the sum of the parallel sides × perpendicular height.

Trapezium, found by dividing into two triangles.

Circle = diameter squared × 0.7854; or, = circumference squared × 0.07958.

Sector of circle = length of arc × half radius.

THE THEORY AND PRACTICE OF MECHANICS

CHAPTER I

KINEMATICS

1. Classification.—The present extent of mechanics makes it desirable, especially for purposes of instruction, to divide the subject into several branches. The following is the customary method of division:

Mechanics
- **Kinematics**; Geometry of motion, excluding the ideas of force and matter.
- **Dynamics**; Action of force upon matter.
  - **Kinetics**; Force and motion.
  - **Statics**; Forces in equilibrium.

It is also convenient to subdivide dynamics according to the nature of the bodies considered. The following is the customary classification, the names on the right being those commonly found in textbooks:

Dynamics of
- Particles; Rigid bodies; — Theoretical or analytical mechanics.
- Elastic bodies; Applied mechanics or strength of materials.
- Fluids; Hydrodynamics or hydraulics.
- Gases; Thermodynamics and aërodynamics.

2. Motion.—Mechanics is concerned primarily with the action of force upon matter. Many problems, however, relate simply to descriptions of motion, without reference to the cause of motion or the nature of the body moved. This has given rise to a special branch of the subject called **Kinematics** (Greek, "pertaining to motion"), which was the first to receive scientific development, and offers a natural introduction to the study of the entire subject.

Since motion involves two quantities, time and space, kinematics may be described as the geometry of motion. The fundamental concepts upon which kinematics is based are thus the intuitive ideas of time and space. The units in terms of which these quantities are expressed are the ordinary units of length and time in common use. A discussion of the standards from which they are obtained is given in Chapter II.

The position of a body is completely determined when the positions of three of its points, not in the same straight line, are given. In general, therefore, problems in motion may be reduced to the study of the motion of mathematical points. This necessitates some means of locating points, such as the rectangular or polar system of coördinates. Such a coordinate system may then be used as a frame of reference upon which the paths followed by any number of moving points, or other characteristics of their motion, may be mapped out and their relations determined.

The path followed by a moving point must be continuous, since it is obviously impossible for a body to disappear for any interval of time, however small, and then reappear and resume its motion. The curvature of the path must also be continuous, except possibly at points where some new factor suddenly enters to affect the motion, in which case there are two or more separate and successive states of motion to be considered.

3. Scalars and Vectors. — In what follows it will be necessary to consider two kinds of quantities: those which involve magnitude only, and those which involve both magnitude and direction. Quantities involving magnitude only, such as temperature, volume, density, work, energy, etc., may be represented by numbers on a scale, and are therefore called **scalars**. Those which involve direction as well as magnitude, such as displacement, velocity, force, moment, etc., are called **vectors**, or steps. A scalar, then, is simply a numerical magnitude, whereas a vector may be represented by a straight line, since it involves direction as well as magnitude.

Scalars are added or subtracted by simply taking the sum or difference of the numbers representing them. To add two or more vectors, however, means to take successively the steps represented by them. Thus in Fig. 1, if AB represents one

vector and BC another, these two steps may be replaced by a single step AC, which accordingly represents their geometric sum or **resultant.** Reversing the process, a single vector AC may be resolved into two or more vectors CD and BC, called its **components.**

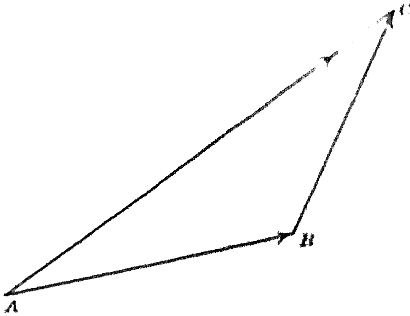


FIG. 1

To add several vectors, the first two, AB and BC (Fig. 2), may be combined into a resultant R_1, this combined with the third vector CD into a resultant R_2, etc., the last resultant obtained being the geometric sum of all those previously added. Or what amounts to the same thing, the vectors AB, BC, CD, etc., may be laid off successively in order, each starting from the terminal point of the one preceding, in which case the final resultant is simply the closing side of the vector polygon so formed, taken in the direction from the initial to the terminal point (Fig. 3). If the initial and terminal points of the series of vectors coincide, it is evident that their resultant is zero.

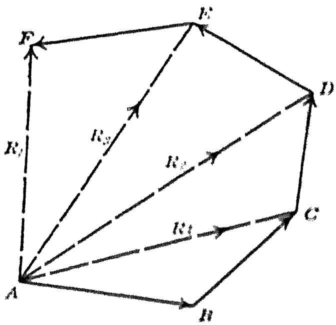


FIG. 2

This construction is called the **vector polygon**, and will be applied in what follows to the composition and resolution of displacements, velocities, accelerations, forces, moments, sectorial velocities, and other vectors.

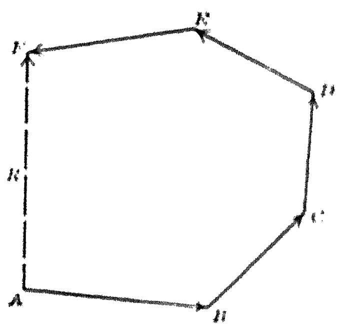


FIG. 3

Note that in a vector polygon the arrows indicating the direction of the vectors are directed from the initial toward the terminal point, whether the path followed is that indicated by the component vectors

or by their resultant. The direction of the resultant regarded as a side of the force polygon therefore always appears to be opposed to that of its components.

4. Composition and Resolution of Displacements. If a motion consists of several successive displacements, it is always equivalent to a single displacement represented by the line joining the initial and terminal points. Thus if in passing from one point *A* to another *B* the path consists of a succession of straight or curved lines, there is always a single path leading directly to the same destination; namely, the straight line joining *A* and *B*. In other words a displacement is a vector quantity, and the resultant of two or more displacements may therefore always be represented by the closing side of a vector polygon, as explained above.

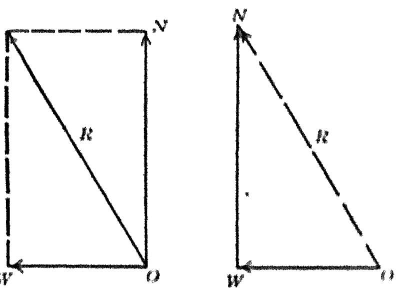


Fig. 4

The same method of composition holds whether the displacements take place successively or simultaneously. To illustrate, suppose a vessel is steered due north by compass and at the same time is carried directly west by the current. The two motions will then take place simultaneously, with the result that the course of the vessel will be west of north. Evidently the resultant displacement would be the same in magnitude and direction, whether the component displacements are considered as taking place simultaneously or successively (Fig. 4). From the same method of reasoning it follows that the resultant of any set of vector displacements may be found by laying them off in succession to scale and then drawing the closing side of the vector polygon so formed (Fig. 5).

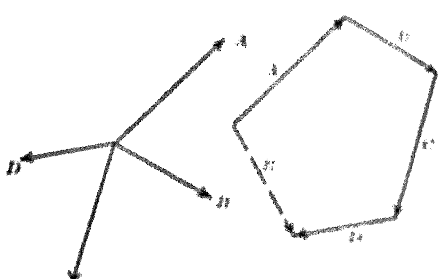


Fig. 5

By reversing the process, any given displacement may be resolved into two or more components by simply drawing a succession of lines, or vectors, which will form with the given vector displacement a closed polygon. This affords a convenient method for finding analytically the resultant of any given set of vector displacements. Thus let a rectangular coördinate system be chosen and resolve each displacement into components parallel to the axes. Then the sum of the components parallel to either axis will be the component of the resultant parallel to that axis.

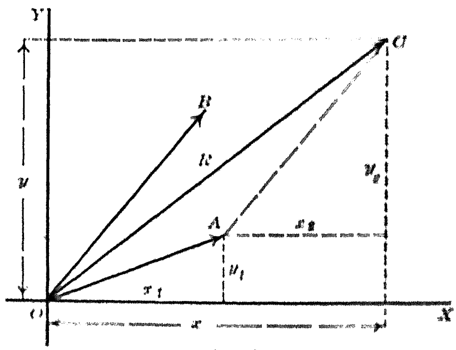


FIG. 6

For instance, in Fig. 6 let OA and OB represent two vector displacements lying in the same plane, and OX, OY the axes of coördinates. Then if x_1, y_1 denote the rectangular components of OA, x_2, y_2 of OB, and x, y of their resultant R, we have

$$x = x_1 + x_2, \quad y = y_1 + y_2;$$

that is to say, the X and Y components of the resultant R are equal to the sums of the X and Y components respectively of OA and OB. The length of the vector R is then found from the relation

$$R = \sqrt{x^2 + y^2},$$

and its direction from $\quad \theta = \tan^{-1}\dfrac{y}{x}.$

Although more indirect than the graphical method of the vector polygon, the analytical method of resolving each displacement into its components parallel to the axes and then finding the resultant from the sums of these several components is more convenient to apply in practice.

PROBLEM

1. When a farm survey is made, the length of each side is measured, and its bearing, or angle with a north and south line, is taken. The following are the survey notes of a five-sided field:

Station	Bearing	Lengths, Feet
1	N. 31½° E.	274
2	N. 85° E.	128
3	S. 58¾° E.	290
4	S. 31¼° W.	353
5	N. 58¼° W.	429

To plot such a survey the method commonly used is to resolve each side into a north and south component, and an east and west component, by multiplying its length by the sine and cosine of its bearing. A meridian is then laid off through the most westerly station, and the components laid off along this meridian and perpendicular to it respectively. Those laid off along the meridian are called *latitudes*, and those perpendicular to it are called *departures*.

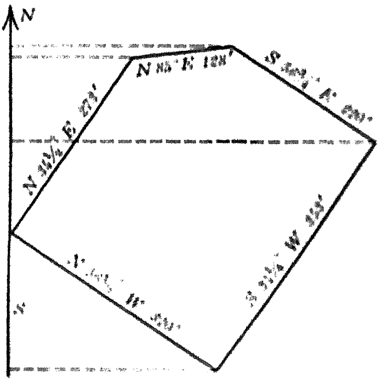


Fig. 7

It is evident that when a person has gone entirely around the field, he has gone as far north as south, and as far east as west. Consequently if the survey is correct, the sum of the north latitudes must equal the sum of the south latitudes, and the sum of the east departures must equal the sum of the west departures. This is equivalent to saying that if the vector polygon closes, the resultant is *zero*, and hence its components are also zero (Fig. 7).

If a considerable error is made in measuring any side, it will be shown by the fact that the algebraic sums of the latitudes and departures are not zero, which is equivalent to saying that the polygon does not close. In this case if the difference between the east and west departures is denoted by x, and between the north and south latitudes by y, the error in measurement is approximately

$$e = \sqrt{x^2 + y^2}$$

and occurs in the course having the direction

$$\theta = \tan^{-1}\frac{y}{x},$$

thus affording a means by which a careless mistake may frequently be located.

In the present problem determine whether or not the survey closes by computing the latitudes and departures, and then balancing the north and south latitudes and east and west departures.

5. Speed and Velocity. — To describe the motion of a point it is necessary to give not only the path it traverses, but also its rate of travel, called the **speed.** If equal distances are passed over in equal times, the motion is said to be **uniform.** In this case the speed is the distance passed over in a unit of time, say one second, and is obtained by dividing the length of any portion of the path by the interval of time in which it was described. Thus for uniform motion, if s denotes the distance described in the time t, the speed v is given by

$$v = \frac{s}{t}.$$

From this definition it is evident that speed is a scalar quantity.

If the distances passed over in equal intervals of time are not equal, the motion is called **non-uniform.** In this case the quotient of distance by time gives merely the average speed of the point in the interval considered. The actual speed at any instant is the limit of this ratio as the interval of time is indefinitely diminished. Thus, in the notation of the calculus, if Δs represents the distance passed over in an interval of time Δt, the speed v at the instant in question is given by

$$v = \lim_{\Delta t \doteq 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

The former definition, $v = \frac{s}{t}$, is a special case of the one just obtained. For in the case of uniform motion, v is constant, and hence writing the general definition, $v = \frac{ds}{dt}$, in the form $ds = v\,dt$, it may, in this case, be integrated directly, giving $s = vt + c$. If the motion is assumed to start at the origin, then $s = 0$ when $t = 0$, and hence $c = 0$. In this case $s = vt$, or $v = \frac{s}{t}$, as obtained above.

If the units of distance and time are chosen as the foot and second, the unit of speed from the above definition will be the foot per second, usually abbreviated into ft./sec.

Velocity differs from speed in that it involves the direction of motion as well as its rate. Thus, in Fig. 8, if a point is moving along a curve, the direction of its motion at any point A is given by the tangent AB to the curve at this point. If, then, the length of this tangent is laid off to scale to represent the speed, the vector AB gives both the direction and rate of motion at the given instant, and therefore represents the velocity of the point at this instant.

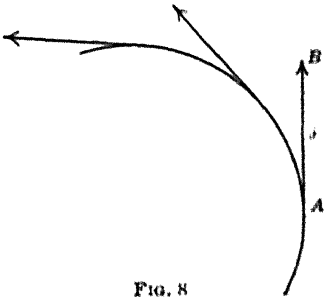


FIG. 8

From this definition of velocity, it is evident that speed is represented by the scalar part of the velocity vector; that is, by its length. Hence constant speed does not imply constant velocity, for the length of the vector AB may remain the same, and yet its direction be constantly altered. For instance, if a body revolves uniformly in a circle, like a point on the rim of a flywheel, or the balls of a simple Watt governor, the speed may be constant, but the velocity is not, since it alters its direction from point to point.

In what follows, velocity will be distinguished from speed where necessary by denoting it by $\bar{v}$; that is, by placing a line above the v to indicate that it is a vector quantity. The same notation will be followed throughout, a line being used to distinguish a vector from a scalar quantity.

PROBLEMS

2. A man shouts and hears the echo one second later. If the velocity of sound is 1100 ft./sec., how far away is the object which produces the echo?

3. In a thunderstorm the clap was heard 5 sec. after the flash was seen. How far away was the discharge?

4. A passenger in a train counts the clicks made by the car wheels in passing over the rail joints, and finds that the number is 45 per minute. If the rails are 100 ft. long, find the speed of the train.

5. A train on the N. Y. C. & H. R. R. R. makes the trip from New York to Albany, 160 mi., in 180 min. Find its average speed.

6. The distance from Queenstown to Sandy Hook is 2779 nautical miles. How long will it take a vessel having an average speed of 25 knots to make the trip? (The knot = 1 nautical mile per hour.)

7. An engine makes 150 r. p. m. If the length of stroke is 18 in., find the piston speed in ft. / min.

8. The speed of a moving point is given by the relation $v = c\sqrt{s}$, where c denotes a constant. Find the relation between the distance and the time.

9. A point describes the diameter of a circle with a speed proportional to the corresponding ordinate. Find the law of distance.

10. The minute hand of a clock is 9 in. long. Find the velocity of its extremity at a quarter past three.

11. If the speed of light is 984,000,000 ft. / sec., how long will it take a ray of light to reach the earth from the nearest fixed star, Alpha Centauri, distant 255,000 times as far as the sun?

NOTE. — The average star distance is about twice as great as that for Alpha Centauri.

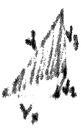
6. Composition and Resolution of Velocities. — If a point moves in a straight line with velocity v for a time t, the effect is to produce a displacement s such that $s = vt$. If, then, the motion of a point is made up of several independent velocities v_1, v_2, v_3, etc., an equal number of displacements s_1, s_2, s_3, etc., would be produced in any given interval of time t, each parallel to the corresponding velocity vector and proportional to it in length, since

$$\frac{s_1}{v_1} = \frac{s_2}{v_2} = \frac{s_3}{v_3} = \cdots = t.$$

These several displacements, however, may be combined into a single resultant displacement by means of a vector polygon, as explained in Art. 4. But since the velocity vectors are parallel and proportional in length to the corresponding displacements, they will also form a vector polygon similar to the displacement polygon. The closing side of this polygon is therefore parallel and proportional to the resultant displacement, and consequently must represent the resultant velocity. It is evident, therefore, that velocities may be combined by means of a vector polygon in the same manner as displacements.

If the polygon formed by the velocity vectors closes, the resultant velocity is zero. If, then, these velocities are considered as existing simultaneously, the closing of the vector polygon means that they completely neutralize or annul one another. For instance, if a boat moves upstream at the same rate that the current is carrying it down, it will appear from the shore to be stationary, the reason being that the two velocities are exactly equal in amount and opposite in direction.

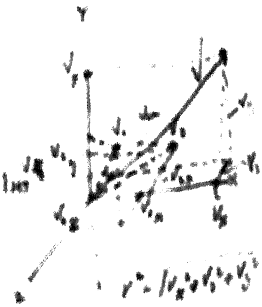
A velocity may be resolved into two or more components by drawing a succession of vectors which will form with the given velocity vector a closed polygon. As in the case of displacements, it is often convenient to resolve a velocity into components by projecting it upon the coördinate axes. For instance, to combine several velocities $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, etc., analytically, first find the projections of each upon the three coordinate axes, the components of $\bar{v}_1$ being denoted by v_{1x}, v_{1y}, v_{1z}; of v_2 by v_{2x}, v_{2y}, v_{2z}, etc. Then the sum of these components along any one axis is equal to the component of the resultant along this axis; that is, if v_x, v_y, v_z denote the components of the resultant v, then



$$v_x = v_{1x} + v_{2x} + v_{3x} + \cdots$$

$$v_y = v_{1y} + v_{2y} + v_{3y} + \cdots$$

$$v_z = v_{1z} + v_{2z} + v_{3z} + \cdots$$



from which the length of the resultant is found to be

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2},$$

and its inclinations to the axes are given by

$$\theta_x = \cos^{-1}\frac{v_x}{v}; \quad \theta_y = \cos^{-1}\frac{v_y}{v}; \quad \theta_z = \cos^{-1}\frac{v_z}{v}.$$

Problems in curvilinear motion may often be simplified by choosing the axes of reference OX and OY in the direction of the tangent and normal to the path at the point in question, as explained in Art. 22. Each velocity is then resolved into tangential and normal components, the sums of which are respectively the tangential and normal components of the resultant velocity.

PROBLEMS

12. The table of a planer has a speed of 40 ft./min. and the tool has a speed at right angles to this motion of 10 ft./min. Find the direction and speed with which the tool will move over a piece of work clamped to the table.

13. A screw thread having 8 threads to the inch is to be cut in a lathe on a rod one inch in diameter. If the speed of the lathe is 30 r. p. m., how fast must the tool advance parallel to the rod?

7. Acceleration. — If the velocity of a moving point varies with the time, the rate of change of velocity is called the **acceleration.** Since velocity is a vector quantity, it may change either in amount or direction.

In the case of non-uniform motion in a straight line, the direction of the velocity vector remains unchanged, but its length varies; as, for instance, in the motion of a street car, which is constantly changing its speed to accommodate the traffic.

If the point moves in a curved path, the velocity is constantly changing in direction, since the direction of motion at any instant is always tangential to the path at the point in question. Curvilinear motion is therefore always accelerated, whether the speed varies or not. For example, the motion of the earth around the sun has an acceleration directed towards the sun, which constantly changes the direction of the earth's motion, causing it to traverse an elliptical orbit instead of moving off in a straight line.

Since change in any vector, such as velocity, must take place in a certain direction as well as have a definite numerical value, acceleration is also a vector quantity. When it is desired to indicate this in what follows, it will be denoted by $\mathbf{a}$.

The numerical value of the acceleration is the rate at which the velocity is changing. If this change is uniform, the rate is found by dividing the change in speed, $v_2 - v_1$, in any given interval of time $t_2 - t_1$ by this interval, giving

$$a = \frac{v_2 - v_1}{t_2 - t_1}.$$

If the point starts from rest and attains a speed v in the time t, this relation becomes simply $a = \frac{v}{t}$, or

$$v = at.$$

If the change is not uniform, the quotient $\frac{v_2 - v_1}{t_2 - t_1}$ gives the average numerical (or scalar) acceleration during the given interval of time. In this case the actual acceleration at any instant is the limit of this quotient as the interval of time is indefinitely diminished. If then $\Delta v = v_2 - v_1$ denotes the change of speed in the interval of time $\Delta t = t_2 - t_1$, the acceleration is given by $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$, or

$$a = \frac{dv}{dt}.$$

Since $v = \frac{ds}{dt}$, this relation may also be written

$$a = \frac{d^2s}{dt^2}.$$

If the change in velocity is uniform, the acceleration a is constant. In this case if the above expression is written in the form $dv = a\,dt$, it may be integrated at once into

$$v = at + c.$$

If the point starts from rest, then $v = 0$ when $t = 0$, and consequently $c = 0$. In this case $v = at$, or $a = \frac{v}{t}$, which agrees with the definition previously given.

If the units of distance and time chosen are the foot and second, then since the unit of speed is the foot per second, the unit of acceleration will be the change in speed in feet per second in each second, usually written feet per second per second, and abbreviated into ft./sec.^{2}.

PROBLEMS

14. A street car in passing over a city block 400 ft. long is first uniformly accelerated and then uniformly retarded, stopping at both ends. If the total time of motion is 20 sec., find the greatest speed attained.

15. A body moving with a velocity of 40 ft./sec. is struck so as to cause it to move off at right angles with a velocity of 60 ft./sec. Find the direction and amount of the acceleration given to it.

16. The earth moves around the sun in an elliptical orbit which is nearly circular. Assuming that the orbit is truly circular and the speed constant, determine whether or not the motion is accelerated, and if so, determine the direction of the acceleration.

8. The Hodograph. — Consider a point moving in a curvilinear path, and let $\bar{v}_1$, $\bar{v}_2$ represent the velocities of the point at two successive positions P_1, P_2 in the path (Fig. 9). Then the change in velocity, or vector acceleration, is represented by the closing side of the vector triangle formed on $\bar{v}_1$, $\bar{v}_2$ as adjacent sides.

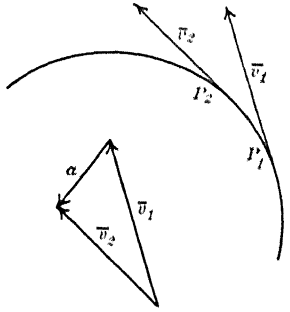


FIG. 9

Now suppose that during any motion of a point the velocities at each instant are all laid off from the same fixed center O (Fig. 10). Then the locus of their outer extremities will be a continuous curve, called the **hodograph**. From this construction it is obvious that to any portion P_1P_2 of the path there corresponds a portion $P_1'P_2'$ of the hodograph, such that the tangent to the hodograph at any point gives the direction of the change in velocity at the corresponding point of the path. Moreover, since any portion of the hodograph, say $P_1'P_2'$, is described in the same length of time as the corresponding portion, P_1P_2, of the path, the rate at which the point P_1' describes the hodograph is equal to the rate of change of the velocity of the corresponding point P_1 in the path. This, however, is the acceleration of P_1 in the path. Hence, the velocity of P_1' at any point of the hodograph represents the acceleration of the actual motion at the corresponding point P_1 of the path.

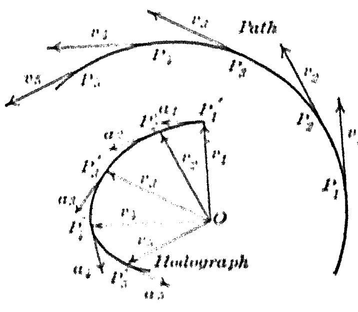


FIG. 10

The importance of the hodograph consists in the fact that it gives a means of representing accelerations as velocities, and thereby considerably simplifies certain problems in motion, as illustrated in Art. 9 and 10.

PROBLEMS

17. Find the hodograph for a point moving with varying speed in a straight line.

18. Show that the hodograph of a projectile is a vertical line.

9. Composition and Resolution of Accelerations. Since by means of the hodograph accelerations may be represented as velocities, the same law of geometric addition applies to both. Hence the resultant of any given set of vector accelerations may be found by means of a vector polygon, as in the case of velocities or displacements. It also follows that a vector acceleration may be resolved into two or more components by simply constructing a triangle or closed polygon which shall have the given vector as one side.

The resultant of any given system of vector accelerations may also be found analytically, as in the case of velocities or displacements, by first resolving each acceleration into rectangular components. Denoting the sums of these components parallel to the three coördinate axes by a_x, a_y, a_z, respectively, the numerical amount of the resultant acceleration is given by

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2},$$

and its inclinations to the axes by

$$\theta_x = \cos^{-1}\frac{a_x}{a}; \quad \theta_y = \cos^{-1}\frac{a_y}{a}; \quad \theta_z = \cos^{-1}\frac{a_z}{a}.$$

In the case of curvilinear motion, it is convenient to choose the coördinate axes tangent and normal to the path. The sums of the components of the given accelerations in these directions will then be the tangential and normal components of the resultant vector acceleration, and may frequently be used in this form instead of recombining them to find the actual resultant.

PROBLEMS

19. A point has three independent accelerations, of 12 ft./sec.², 9 ft./sec.², and 15 ft./sec.², inclined at angles of 120° to one another. Find the direction and amount of the resultant acceleration.

20. A point describes a parabola $y^2 = 4px$ with uniform speed c. Find the component of the acceleration parallel to the axis of the parabola.

21. A planet moves in an elliptical path about the sun as a focus. If at a given instant the radius vector makes an angle of 65° with the tangent to the path, and the acceleration toward the sun is denoted by a, compare the tangential and normal components of this acceleration. Also determine at what points the tangential component of the acceleration is least, and how this affects the speed at these points.

10. Central Acceleration. — As previously mentioned, it is often convenient to resolve the acceleration into components tangential and normal to the path. Since the normal component is perpendicular to the tangent, it is directed toward the center of curvature. The effect of the normal component of the acceleration is, then, to alter the direction of the motion, whereas the effect of the tangential component is to alter the speed.

In Art. 7 the tangential component a_t of the acceleration was found to be

$$a_t = \frac{d^2s}{dt^2},$$

where s denotes the distance traversed along the path.

The normal component of the acceleration may be found by considering uniform motion in a circle, since the actual motion in the path at any instant is equivalent to motion in the osculating circle, that is, a circle whose center coincides with the center of curvature of the path at the point in question.

Consider, then, a point moving with uniform speed v in a circle of radius r. In this case the hodograph is evidently a circle of radius v, which is described in the same time as the path. Consequently the speeds with which these two circles are described are proportional to their radii. Therefore, since the speed with which the hodograph is described is the required acceleration, say a_n, we have $a_n : v = v : r$, or

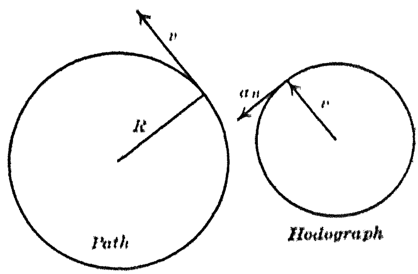


FIG. 11

$$a_n = \frac{v^2}{r}.$$

Moreover, it is evident from Fig. 11 that a_n is perpendicular to v and therefore parallel to r; that is to say, it is directed toward the center of curvature, or normal to the path.

PROBLEMS

22. A pail of water is swung in a vertical circle by a string attached to it. What velocity must the pail have at the highest point in order that the water may remain in it?

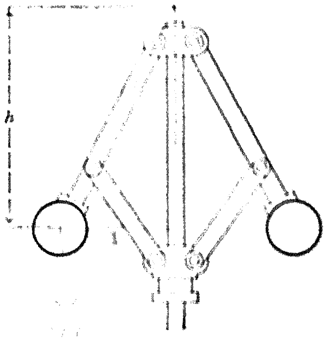


FIG. 12

23. A conical pendulum, such as a simple Watt governor on a steam engine (Fig. 12) makes 120 r.p.m. Find its height h, i.e. the distance from the center of suspension to the plane in which the balls revolve.

24. Show that the height of a conical pendulum depends only on the speed and is independent of the length of the arm.

25. In a circle swing the cars hang 15 ft. from the mast or vertical axis around which they revolve and are suspended by cables 20 ft. long. If they swing out when in motion until the cables make an angle of 30° with the vertical, what is the greatest speed attained?

26. What velocity would it be necessary to give a bullet in order that it should not fall to the earth but encircle it continually like a satellite? Assume the radius of the earth to be 4000 mi.

11. Analytical Derivation.

The expression for the normal acceleration obtained in the preceding article may be derived analytically as follows:

Consider a plane path and let a_x and a_y denote the components of the acceleration parallel to any pair of rectangular axes in the same plane as the path. Then from Fig. 13, the normal acceleration a_n is given by

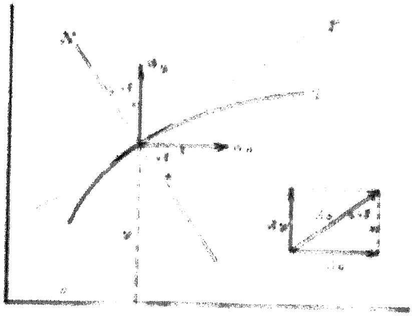


FIG. 13

$$a_n = a_y \sin\alpha - a_x \cos\alpha,$$

or, since $\sin\alpha = \dfrac{dx}{ds}$ and $\cos\alpha = \dfrac{dy}{ds}$, this becomes

$$a_n = a_y\frac{dx}{ds} - a_x\frac{dy}{ds}.$$

Furthermore, since $\dfrac{dx}{ds} = \dfrac{dx}{dt}\cdot\dfrac{dt}{ds} = v_x\dfrac{dt}{ds}$, and $\dfrac{dy}{ds} = v_y\dfrac{dt}{ds}$, this may be written

$$a_n = (a_y v_x - a_x v_y)\frac{dt}{ds}.$$

It is shown in the calculus, however, that when the coördinates x, y of any point of a curve are expressed in terms of a single parameter t, the radius of curvature r of the curve at the point x, y is given by the expression

$$r=\frac{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}=\frac{\left(\frac{ds}{dt}\right)^3}{v_xa_y-v_ya_x}.$$

Therefore eliminating $v_xa_y-v_ya_x$ between these two expressions, we have finally

$$a_n=\frac{\left(\frac{ds}{dt}\right)^2}{r}=\frac{v^2}{r},$$

as the general expressions for the normal acceleration in any curved path in terms of its radius of curvature.

12. Uniformly Accelerated Motion. — In Art. 7 the fundamental equation for rectilinear motion was found to be

$$a=\frac{d^2s}{dt^2}.$$

When the acceleration a is constant, the motion is said to be uniformly accelerated. In this case, integrating this relation with respect to the time t we have

$$\frac{ds}{dt}=at+c_1,$$

where c_1 denotes a constant of integration. Integrating again,

$$s=\tfrac{1}{2}at^2+c_1t+c_2,$$

where c_2 is also a constant of integration. Since $\frac{ds}{dt}=v$, the first integral may be written

$$v=at+c_1.$$

To determine the constants of integration let v_0 denote the initial velocity, that is, at the time $t=0$, and s_0 the initial distance of the point from the origin. Substituting these simultaneous values, namely, $s=s_0$ for $t=0$ and $v=v_0$ for $t=0$, in the above

relations, the constants of integration are found to be $c_1 = v_0$, $c_2 = s_0$. Consequently the above relations become

$$v = v_0 + at, \tag{1}$$

$$s = s_0 + v_0 t + \tfrac{1}{2} at^2. \tag{2}$$

Since the origin of coördinates is arbitrary, it may be so chosen that $s_0 = 0$. If the point starts from rest, v_0 is also zero, and the formulas simplify into

$$v = at, \tag{3}$$

$$s = \tfrac{1}{2} at^2. \tag{4}$$

By eliminating the time t between Eq. (1) and (2) another useful relation may be obtained, namely,

$$v^2 = v_0^2 + 2a(s - s_0). \tag{5}$$

If s_0 is zero, this becomes $v^2 = v_0^2 + 2as$,

and if the initial velocity v_0 is also zero, it further simplifies into

$$v^2 = 2as. \tag{6}$$

This relation may also be obtained directly from the definitions of velocity and acceleration by multiplying together the corresponding members of the two expressions

$$v = \frac{ds}{dt}, \text{ and } \frac{dv}{dt} = a,$$

which gives

$$v\frac{dv}{dt} = a\frac{ds}{dt},$$

or, omitting the common factor dt,

$$v\,dv = a\,ds,$$

which for uniform velocity, that is, a = constant, integrates into Eq. (5).

PROBLEM

27. A particle is projected with initial velocity v_0 at an angle θ to the horizontal (Fig. 14). Neglecting the resistance of the air, find the equation of the path or trajectory, the maximum height attained, the range, and the angle of elevation for maximum range.

SOLUTION. The equations of motion in this case are

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g,$$

the integrals of which are

$$x = c_1t + c_2;$$

$$y = -\tfrac{1}{2}gt^2 + c_3t + c_4.$$

From the initial conditions the values of the constants of integration are found to be

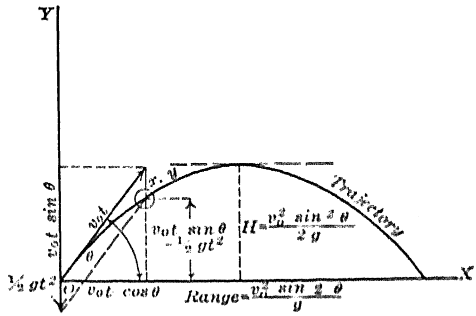


FIG. 14

$$c_1 = v_0 \cos\theta; \quad c_2 = 0; \quad c_3 = v_0 \sin\theta; \quad c_4 = 0.$$

Substituting these values and eliminating t, the equation of the trajectory becomes

$$y = x \tan\theta - \frac{gx^2}{2\,v_0^2 \cos^2\theta}. \tag{7}$$

The greatest height H is obtained from the condition $\frac{dy}{dx} = 0$, whence

$$H = \frac{v_0^2 \sin^2\theta}{2\,g}. \tag{8}$$

The range is found by putting $y = 0$ in the equation of the trajectory and solving for x, whence

$$\text{Range} = \frac{v_0^2 \sin 2\,\theta}{g}. \tag{9}$$

The angle of elevation for maximum range is found from the condition $\frac{dx}{d\theta} = 0$, whence $\cos 2\,\theta = 0$ and consequently $\theta = 45°$. Substituting this value of θ in Eq. (9) we have

$$\text{Maximum range} = \frac{v_0^2}{g}. \tag{10}$$

When the resistance of the air is taken into account, the problem becomes more complicated. The equation of the trajectory in this case has been determined empirically by Hélie as

$$y = x \tan\theta - \frac{gx^2}{2\cos^2\theta}\left(\frac{1}{v_0^2} + \frac{kx}{v_0}\right), \tag{11}$$

which is a modification of Eq. (7) obtained by the introduction of an empirical constant k, given by the formula

$$k = 0.0000000458\,\frac{d^2}{w},$$

where d is the diameter of the projectile in inches and w is its weight in pounds. From this formula a range table may easily be constructed.

28. In the subway of the Interborough Rapid Transit Co. under the East River, between New York and Brooklyn, the grade is 3.1 per cent. The profile of the south tunnel, Brooklyn to Manhattan, is shown in Fig. 15, the speeds given being for the fast express service. The length of the safety stop for

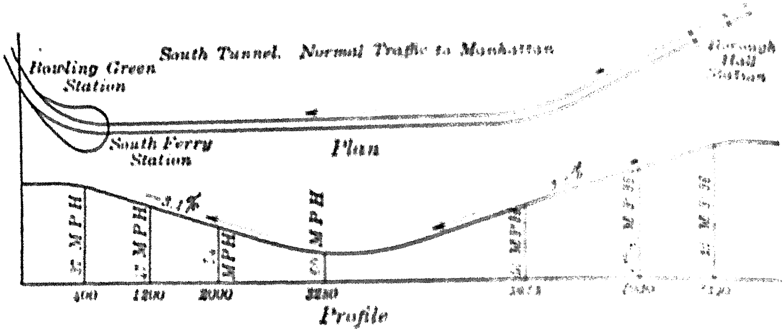


FIG. 15

each block is assumed as 50 per cent in excess of the actual braking distance. Given that the length of the safety stop for the speed of 25 mi. hr. at the top of the grade is 600 ft., find the lengths of the safety stops for each of the other speeds indicated on the profile.

29. A train in starting attains its full speed of 60 mi. hr. in 5 min. After running for a certain length of time at this speed, the brakes are applied and it stops in 4 min. If the total distance is 10 mi., find the total time.

30. A steamship approaches a wharf with uniformly retarded motion. If it moves 200 ft. in the first 10 sec. of the retarded motion and 50 ft. in the next 10 sec., how long will it be before it will stop?

31. In launching a ship, the time of sliding down the ways, a distance of 300 ft., was 10 sec. After entering the water the ship experienced a uniform resistance, and stopped after moving a distance of 1200 ft. from the foot of the ways. Find the total time.

32. An automobile reduces its speed from 40 mi. hr. to 30 mi. hr. in 5 sec. If the retardation is uniform, how much longer will it be before it will come to rest, and how far will it travel in this length of time?

13. Graphical Representation. — The relations between distance, speed, acceleration, and time may be represented graphically by laying off these quantities along rectangular axes and drawing curves through the points so located, as explained in Art. 14–17. In solving simple problems in uniform motion, the solution is most easily obtained by the use of the formulas derived in Art. 12. In more complex cases, especially when the law

governing the motion is to be determined from observed data obtained by experiment, the graphical method is often preferable.

The graphical method also has the advantage of showing at a glance the nature of the entire motion, thus frequently bringing out facts which might otherwise be overlooked.

14. Distance-time Curve. — To represent the motion of a point graphically, construct a pair of rectangular axes and lay off the distances traversed by the point since starting along one axis, say OY (Fig. 16), and the corresponding times along the other axis, OX. The locus of the points so obtained will then be a curve (or straight line) called the **distance-time curve.** Note that the distance-time curve does not represent the path followed by the moving point, but merely exhibits the relation between the space passed over and the time consumed.

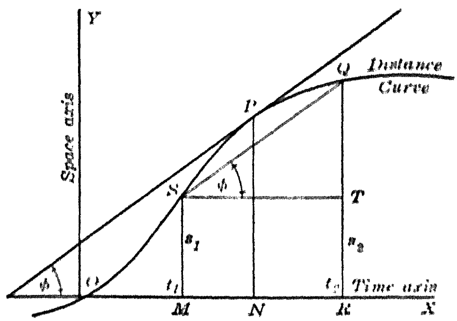


FIG. 16

At the time OM, or t_1, the distance traversed since starting is MS, or s_1; at the time OR, or t_2, the distance is RQ, or s_2, etc. Consequently in the interval of time MR, or $t_2 - t_1$, the distance passed over is QT, or $s_2 - s_1$, and therefore the average speed during this interval of time is

$$v_{\text{average}} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{QT}{ST} = \tan \phi.$$

As the interval of time ST is indefinitely diminished, the chord SQ approaches tangency, and the average speed approaches the actual speed at some point P of the interval considered. Therefore the slope of the tangent to the distance-time curve at any point represents the speed at that particular instant.

This is equivalent to saying that the definition of speed given in Art. 5, namely, $v = \frac{ds}{dt}$, is the differential equation of the distance-time curve. For uniform motion the speed is constant,

and consequently the distance-time curve has the same slope throughout; that is to say, it becomes a straight line.

The fact that the average speed in any interval of time is equal to the actual speed at some instant during this interval is an example of what is known in the calculus as the theorem of mean value.

15. Speed-time Curve. — If the speeds of a moving point are plotted as ordinates and the corresponding times as abscissas, the locus of the points so determined is called the **speed-time curve**.

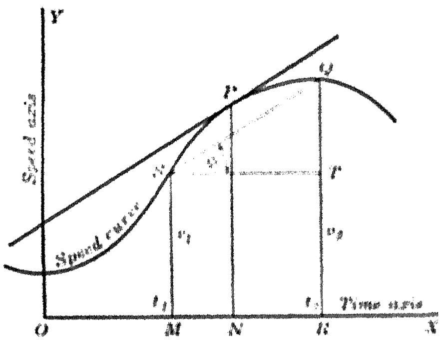


FIG. 17

In Fig. 17, let any given time t_1 be represented by OM, and the speed v_1 at this instant by SM, thus locating a point S of the curve. Similarly, any other point Q is determined by the coordinates t_2, v_2; etc. Hence QT, or $v_2 - v_1$, represents the change in speed in the interval of time MR, or $t_2 - t_1$, and their quotient is the average rate of change of speed in this interval of time, or the average acceleration; that is to say,

$$a_{\text{average}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{QT}{ST} = \tan\phi.$$

As the interval of time ST is indefinitely diminished the chord SQ approaches tangency, and the average acceleration approaches the actual acceleration at some point P of the interval considered. Hence the slope of the tangent to the speed-time curve at any point represents the acceleration at that particular instant.

The definition of acceleration given in Art. 7, namely $a = \dfrac{dv}{dt}$, may also be considered as the differential equation of the speed-time curve.

For uniformly accelerated motion the acceleration is constant, and therefore the speed-time curve becomes a straight line.

16. Acceleration-time Curve. — An acceleration-time curve may be obtained in the same way as the distance and speed curves by plotting the accelerations at given instants as ordinates and the corresponding times as abscissas.

The slope of the acceleration-time curve has no special significance, but the area under the curve has a physical meaning, as explained in what follows.

17. Relations between the Distance, Speed, and Acceleration Curves. — In referring to these curves in the present article, it will be convenient to omit the word "time."

Let the equation of the distance curve be denoted by $s = f(t)$, where $f(t)$ denotes some function of the time. Whether or not the form of this function can be actually determined from the given data is immaterial for the present purpose.

Since $v = \frac{ds}{dt}$, the ordinate v to the speed curve at any instant t is numerically equal to the slope of the distance curve at the same instant. Similarly, since $a = \frac{dv}{dt}$, any ordinate to the acceleration curve is numerically equal to the corresponding slope of the speed curve. The equation of the three curves may then be written

$$s = f(t); \quad v = \frac{ds}{dt} = f'(t); \quad a = \frac{d^2s}{dt^2} = f''(t);$$

where $f'(t)$ and $f''(t)$ denote the first and second derivatives of $f(t)$, respectively.

Since the calculus condition that $f(t)$ shall be a maximum or a minimum is $f'(t) = 0$, it follows that the distance curve has a maximum or minimum point where the speed curve crosses the axis. Since a similar relation exists between $f'(t)$ and $f''(t)$, the speed curve has a maximum or minimum point where the acceleration curve crosses the axis.

Furthermore, the calculus condition that $f(t)$ shall have a point of inflection is $f''(t) = 0$. Consequently at a point where the distance curve has a point of inflection, the acceleration curve crosses the axis.

These relations are illustrated graphically in Fig. 18.

The areas under these curves are also related. Thus writing the relation $v = \frac{ds}{dt}$ in the form $ds = v\,dt$ and integrating between corresponding limits, we have

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt.$$

The left member of this expression, however, is the distance described in the given interval of time, while the right member

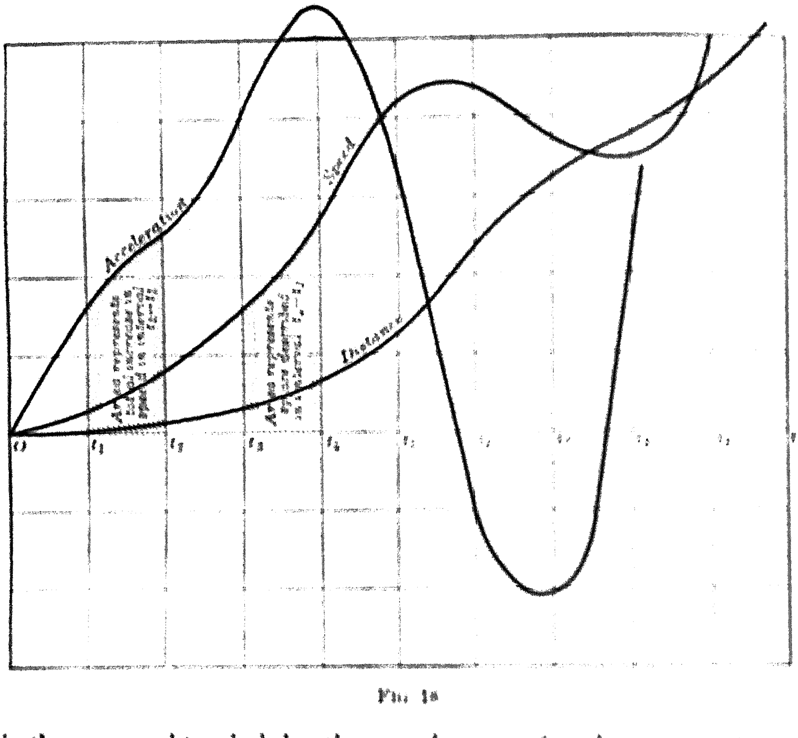


FIG. 18

is the area subtended by the speed curve for the same interval.

That is to say, if ordinates are drawn to the speed curve at the times t_2 and t_p, the area under the speed curve between these ordinates is numerically equal to the distance described in the given interval of time.

Similarly, since $a = \frac{dv}{dt}$ we have $dv = a\,dt$, and integrating be-

tween corresponding limits,

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt.$$

Hence the area under the acceleration curve between any two ordinates drawn at the times t_1 and t_2 represents the increase in speed during this interval of time.

PROBLEMS

33. A train starting from rest covers the distances s feet in t seconds as follows:

t	0	5	12	17	20	26	31	40	45	51	57	65
s	0	25	60	125	200	300	400	590	660	750	805	860

Find the average speeds at intervals of 10 sec., and plot a speed-time curve.

34. From the speed-time curve obtained in the preceding problem determine the acceleration at intervals of 10 sec., and from these results plot an acceleration-time curve.

35. A car starting from rest has speeds of v feet per second after t seconds from starting as follows:

t	0	5	11	16	21	29	34	39	47	53	60
v	0	16	40	49	60	69	68	60	45	27	36

Determine the average acceleration at 5-second intervals, and draw the acceleration-time curve.

36. From the acceleration-time curve obtained in the preceding problem, determine the average speeds at 5-second intervals, and draw the speed-time curve.

18. Falling Bodies. — It has been found by observation and experiment that the motion of a body falling freely toward the earth under the attraction of gravitation is uniformly accelerated. In other words, the acceleration of a falling body, unresisted in its motion, is constant.

The acceleration in this case is denoted by g, the initial letter of the word "gravity." The value of g depends on the distance from the center of the earth, and it is therefore constant only for

a particular place, changing with the latitude and the elevation above sea level. From theory and experiment the value of g has been determined as

$$g = 32.0894(1 + 0.0052375 \sin^2 \phi)(1 - 0.0000000957 E),$$

where ϕ denotes the latitude of the place in degrees, and E is its elevation above sea level in feet. (See Prob. 54, Art. 22.)

For purposes of computation it is more convenient to write this expression in the form

$$g = 32.1734 - 0.084 \cos 2\alpha - 0.000003 E,$$

from which the value of g at sea level in latitude 45° is found to be $g = 32.1734$ ft./sec.^{2}. It is therefore customary in practice to assume as an average value $g = 32.2$ ft./sec.^{2}, or in the metric system $g = 981$ cm./sec.^{2}.

Since a falling body is uniformly accelerated, its equations of motion are obtained from the equations previously deduced for uniformly accelerated motion by substituting g for a. Thus for a body falling from rest we have

$$\begin{cases} v = gt, \\ s = \frac{1}{2} gt^2, \\ v^2 = 2\,gs. \end{cases}$$

PROBLEMS

37. A stone is dropped into a well and the splash is heard 3 sec. afterward. Assuming the velocity of sound to be 1100 ft./sec., how far is it to the surface of the water?

38. A bag of sand is thrown out of a balloon which is rising with an acceleration of 5 ft./sec.^{2}. If it reaches the ground in 6 sec., how high was the balloon when the sand was thrown out?

39. The cable of an elevator breaks when the car is 100 ft. above the bottom of the shaft. With what velocity will it strike the bottom?

40. In deducing the laws of falling bodies, Galileo rolled balls down grooves in inclined planes, and by marking off distances of 1, 4, 9, etc., on the grooves observed that the times of descent through these distances were in the ratio 1, 2, 3, etc. Show that this gives the law $s = \frac{1}{2} gt^2$.

NOTE. — Inclined planes were used in order to retard the motion and thus make the observations more accurate. A modern form of this experiment is found in Atwood's machine, explained in Prob. 88.

19. Inclined Plane. — Consider the motion of a point constrained to move along a plane inclined at an angle α to the horizontal. If acted upon only by gravity, the acceleration is of amount g directed vertically downward, the effective component of which along the plane, that is, tangential to the path, is $g \sin \alpha$.

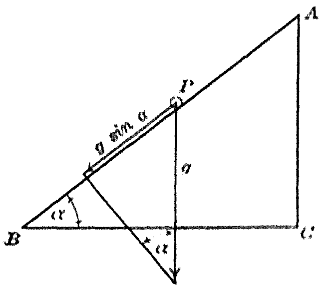


FIG. 19

If then the point starts from A (Fig. 19), its speed when it reaches B is given by

$$v^2 = 2\,as\,;$$

or since in the present case the acceleration a in the direction of motion is $g \sin \alpha$, and $s = AB$, this relation becomes

$$v^2 = 2\,g \sin \alpha AB.$$

Since $AB \sin \alpha = AC$, this may also be written

$$v^2 = 2\,gAC.$$

Therefore, the velocity of the moving point at B is independent of the length of the plane, and is the same as though the particle had fallen freely through the height AC.

This result is a special case of a more general theorem, namely, that whatever the form of the path, the velocity of a particle moving under the action of gravity depends only on the vertical distance fallen. For a demonstration of this theorem see Art. 130.

The time of sliding down the plane AB is found from the equation $s = \frac{1}{2}\,at^2$, where $s = AB$, and $a = g \sin \alpha$. Therefore

$$AB = \tfrac{1}{2}\,g \sin \alpha t^2.$$

From the same relation the time t_1 of falling through the vertical distance AC is given by $\quad AC = \frac{1}{2}\,gt_1^2.$

Dividing one of these equations by the other, substituting $AC = AB \sin \alpha$, and taking the square root, we find that

$$t_1 = t \sin \alpha.$$

Consequently, the time of sliding down the plane AB is greater than the time of falling through the height AC in the ratio $1 : \sin \alpha$, although the speed acquired in each case is the same.

PROBLEMS

41. A body is projected up a plane inclined at 30° to the horizontal with a speed of 88 ft./sec. When and where will it come to rest?

42. A vertical circle has a series of chords radiating from its highest point. Show that the time required for a particle to slide down each chord is the same, and equal to the time of falling down the vertical diameter.

43. Find the straight line down which a particle would slide in the shortest time from a given point A to an arbitrary inclined line BC in the same vertical plane.

HINT. — Make use of the result of the preceding problem.

44. A ball rolls off a roof inclined at 30° to the horizontal. If it starts 15 ft. from the edge of the roof, and the latter is 25 ft. above the ground, find where the ball will strike the ground.

20. Angular Motion. — In moving from one point A to another B, a particle undergoes an angular displacement about any fixed point O, measured by the angle AOB (Fig. 20). For any other center O', the angular displacement is, in general, of different amount. In what follows angular displacement will be denoted by θ, and expressed in radians.

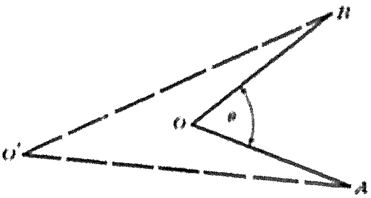


FIG. 20

Angular velocity is defined, as in the case of linear velocity, as the rate of change of angular displacement. It is customary to denote the scalar part of this vector, or angular speed, by ω, in which case

$$\omega = \frac{d\theta}{dt}.$$

ω is here expressed in radians per second, and may be either uniform or varying.

Similarly, angular acceleration is denoted by α, and is defined as the rate of change of angular velocity, whence

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Since ω is expressed in rad./sec., α is expressed in radians per second per second, abbreviated into rad./sec.^{2}.

Since these definitions are precisely similar to those for linear speed and acceleration, the equations of angular motion, or integrals of the above differential equation, will also be similar to those for linear motion. For convenience of comparison and reference, they are given below in parallel columns.

	Linear Motion, Translation	Angular Motion, Rotation
Definitions	$v = \dfrac{ds}{dt}$ $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$ $vdv = ads$	$\omega = \dfrac{d\theta}{dt}$ $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$ $\omega d\omega = \alpha d\theta$
Special equations for initial velocity zero and motion starting at origin	$v = at$ $s = \frac{1}{2}at^2$ $v^2 = 2as$	$\omega = \alpha t$ $\theta = \frac{1}{2}\alpha t^2$ $\omega^2 = 2\alpha\theta$
General equations	$v = v_0 + at$ $s = s_0 + v_0t + \frac{1}{2}at^2$ $v^2 = v_0^2 + 2a(s - s_0)$	$\omega = \omega_0 + \alpha t$ $\theta = \theta_0 + \omega_0t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

PROBLEMS

45. A wheel making 20 rev./sec. is brought to rest in half a minute. How many revolutions did it make after the brake was applied?

46. What is the angular velocity of the earth about its axis?

47. An automobile with wheels 30 in. in diameter is moving at 40 mi./hr If it slows down to 20 mi./hr. in 10 sec., what is the angular retardation of the wheels?

21. Vector Representation.—The method of vector representation used for linear velocities and accelerations may be extended to angular velocities and accelerations. Thus suppose that the radius OA (Fig. 21) rotates about O with angular velocity ω. Then the line OC, drawn through O perpendicular to the plane of the motion, is the axis of rotation. Hence, if the distance OB is laid off along this axis to represent the numerical, or scalar,

value of $\overline{\omega}$, the vector OB completely defines the angular velocity. It is customary to choose the direction of OB as that in which a right-handed screw would advance if turned in the same direction as that in which the radius OA revolves. Rotation in the opposite direction to that shown in Fig. 21 would therefore be indicated by reversing the direction of OB, thus making it point downward along the axis instead of upward.

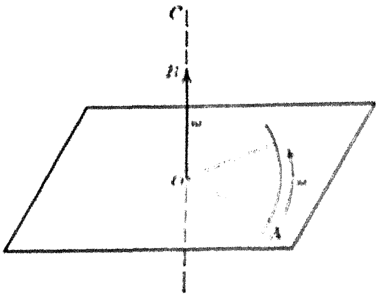


FIG. 21

Similarly, angular acceleration may also be represented by a vector. The resultant of a system of angular velocities or accelerations may, therefore, be obtained by means of a vector polygon, as in the case of linear velocities and accelerations. A proof of the vector polygon construction for combining angular velocities is given in Art. 25.

22. Normal and Tangential Components. Consider a point moving in a circle of radius r with angular speed ω, and let v denote its linear speed along the path, that is, tangential to the circle (Fig. 22). Then, if $d\theta$ denotes the angle described about the center in an interval of time dt, the length of the arc so described is $rd\theta$. Also, since v is the linear speed along the arc, the length of the arc is vdt. Hence, $rd\theta = vdt$, or since $\frac{d\theta}{dt} = \omega$, this relation becomes

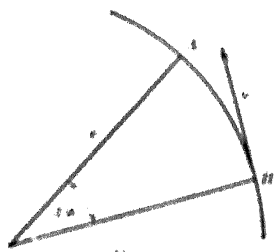


FIG. 22

$$v = r\omega. \tag{12}$$

The linear speed of a point moving in a circle thus depends jointly on its distance from the center and its angular speed about this point.

Differentiating Eq. (12) with respect to the time, we have, since r is constant,

$$\frac{dv}{dt} = r\frac{d\omega}{dt},$$

or, since $\frac{dv}{dt} = a_t$ and $\frac{d\omega}{dt} = \alpha$, this becomes

$$a_t = r\alpha, \tag{13}$$

which gives a similar relation between the angular acceleration and the tangential component of the linear acceleration.

The normal component of the linear acceleration was found in Art. 11 to be $a_n = \frac{v^2}{r}$, or, since $v = r\omega$, this may be written

$$a_n = r\omega^2. \tag{14}$$

The effect of the tangential component a_t is to change the speed along the path, whereas the effect of the normal component a_n is to alter the direction of motion.

For a point moving in a curved path other than a circle, that is, one in which the radius of curvature varies from point to point, the vector velocity may be resolved into two components; a radial component of amount $v_r = \frac{dr}{dt}$, and a transverse component of amount $v_t = r\frac{d\theta}{dt}$. The speed of the point along the path, $v = \frac{ds}{dt}$, is therefore given by the relation

$$\left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2.$$

This relation may also be obtained from the calculus expression for the length of an arc of a curve in polar coördinates. Thus, if r, θ denote polar coördinates, the length of an elementary arc ds is found in the differential calculus to be

$$ds = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta.$$

Dividing both sides by dt, and squaring, we have

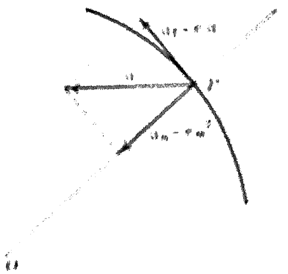
$$\left(\frac{ds}{dt}\right)^2 = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]\left(\frac{d\theta}{dt}\right)^2,$$

which simplifies into

$$\left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2.$$

If the path is a circle, r is constant, and this relation simplifies into $v = r\omega$, as obtained above.

The components of the vector acceleration for a point moving along any curved path are



$$a_x = \frac{d^2x}{dt^2},\quad a_y = \frac{d^2y}{dt^2},\quad a_z = \frac{d^2z}{dt^2}.$$

Consequently, the numerical value of the acceleration is given by

$$a^2 = \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2,$$

Fig. 23

and its direction by

$$\cos(ax) = \frac{\dfrac{d^2x}{dt^2}}{a},\quad \cos(ay) = \frac{\dfrac{d^2y}{dt^2}}{a},\quad \cos(az) = \frac{\dfrac{d^2z}{dt^2}}{a}.$$

It can be shown, however, that this value of a is not in general equal to $\frac{d^2s}{dt^2}$, since $\frac{d^2s}{dt^2}$ represents the acceleration considering only the change in length of the velocity vector and not its change in direction.

The expression for the normal acceleration, $a_n = \dfrac{\left(\dfrac{ds}{dt}\right)^2}{r}$, where r denotes the radius of curvature of the path, is, however, perfectly general, as shown in Art. 11.

PROBLEMS

48. A smooth tube is inclined at an angle ϕ to a vertical axis around which it revolves with angular speed ω. At what point in the tube will a particle come to rest?

49. A weight is swung by a string in a vertical circle 4 ft. in diameter. What must be its speed at the highest point in order that the string may just remain straight?

50. A particle resting upon the highest point of a horizontal cylinder of radius r slides off. Show that it leaves the cylinder after descending a distance of $\frac{r}{3}$, and then describes a parabola with latus rectum $\frac{16r}{27}$.

51. In a centrifugal railway, or "loop the loop," show that the minimum speed at the highest and lowest points of the loop are $\sqrt{gr}$ and $\sqrt{5gr}$, respectively, where r denotes the radius of the loop in feet.

52. If the loop in the preceding problem is 15 ft. in diameter, from what height must the car descend in order to stick to the loop?

53. How much faster than at present would the earth have to revolve in order that a body at the equator should have no weight? Equatorial radius = 20,926,062 ft.

54. Determine the effect of the rotation of the earth on the acceleration due to gravity.

SOLUTION. A place of latitude ϕ describes a circle of radius $r\cos\phi$ (Fig. 24). Hence if ω is the angular speed with which the earth revolves on its axis, the centrifugal acceleration at the latitude ϕ is $r\cos\phi\,\omega^2$. The resultant of this and the actual acceleration a due to gravity is then the apparent acceleration g. Resolve the centrifugal acceleration $r\omega^2\cos\phi$ into tangential and normal components of amounts $r\omega^2\cos\phi\sin\phi$, and $r\omega^2\cos^2\phi$, respectively. Then

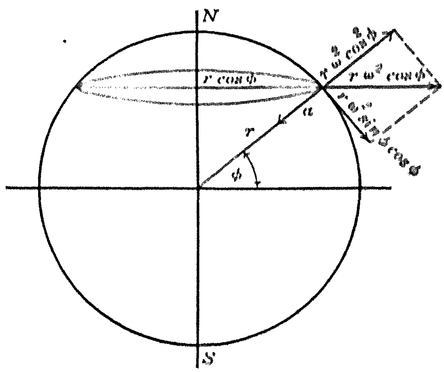


FIG. 24

$$g = a - r\omega^2\cos^2\phi;$$

or, since $\cos^2\phi = 1 - \sin^2\phi$, and $a - r\omega^2 = g_0$, the acceleration at the equator where $\phi = 0$, this relation may be written

$$g = g_0 + r\omega^2\sin^2\phi.$$

Since the period of revolution of the earth is 23 hr. 56 min. 4.09 sec., or 86164.09 sec., its angular speed is

$$\omega = \frac{2\pi}{86164.09} = 0.00007292 \text{ rad./sec.}$$

Moreover, by pendulum experiments the value of g at the equator has been found to be $g_0 = 32.0894$ ft./sec.². Hence assuming the earth to be a sphere of radius $R = 20,900,000$ ft., we have finally $g = 32.0894 + 0.111136\sin^2\phi$, or

$$g = 32.0894\ (1 + 0.003463\sin^2\phi).$$

The fact that the earth is not truly spherical but flattened at the poles changes the value of the constant inside the parenthesis. By means of pendulum experiments it has been found that the actual value of g at sea level is given by the formula

$$g = 32.0894\ (1 + 0.0052375\sin^2\phi).$$

For a point at an elevation E above sea level, this value of g must be corrected by multiplying by the quantity $(1 - 0.0000000957\,E)$, as stated in Art. 18.

23. Moments. — The relation of a vector to the motion of a particle frequently depends on the distance of the vector from some

fixed point. For example the linear speed of a particle rotating about a fixed axis with angular speed ω is given by $v = r\omega$, and therefore depends on the distance r of the particle from the axis of rotation.

This gives rise to a special kind of products called **moments**, where the moment of a vector about any point O is defined as the product of the vector by its perpendicular distance from O.

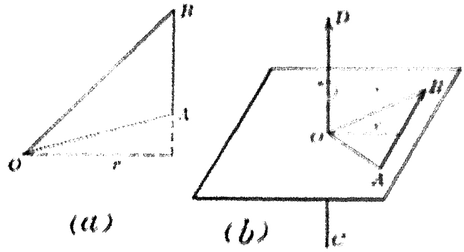


Fig. 25

A moment may be represented graphically in various ways. Thus in Fig. 25 (*a*), if AB represents the vector and O the given point, the moment of AB about O is evidently represented geometrically by twice the area of the triangle AOB.

A moment may also be represented by a vector. Thus in Fig. 25 (*b*), if AB represents the vector part of the moment, O the given point, and AOB the plane of the moment, then if a line CD is drawn through O, perpendicular to the plane of the moment, and of length equal to its numerical value, the moment will be completely represented by the vector CD.

The sign of the moment is indicated by the direction of the vector. The method used for determining the sign is to make the vector CD point in the direction in which a right-handed screw would advance if turned about CD as an axis in the same direction as that indicated by the revolution of AB about O.

24. Fundamental Theorem of Moments.—In Fig. 26, let V_1 and V_2 be any two vectors, and V their resultant. Also let O be any given point, and θ_1, θ_2, ϕ the angles between OA and the vectors V_1, V_2, V, respectively. Then taking moments about O according to the definition of moment given in the preceding article, we have

$$\text{Moment of } V_1 \text{ about } O = V_1 OA \sin\theta_1,$$

$$\text{Moment of } V_2 \text{ about } O = V_2 OA \sin\theta_2,$$

whence by addition

$$V_1 OA \sin\theta_1 + V_2 OA \sin\theta_2 = OA\,(V_1 \sin\theta_1 + V_2 \sin\theta_2)$$
$$= OA \cdot V \sin\phi,$$

since $$V \sin\phi = V_1 \sin\theta_1 + V_2 \sin\theta_2.$$

The last term, however, is the moment of the resultant V with respect to O. Therefore, since O is arbitrary, the sum of the moments of any two vectors with respect to a given point is equal to the moment of their resultant with respect to this point.

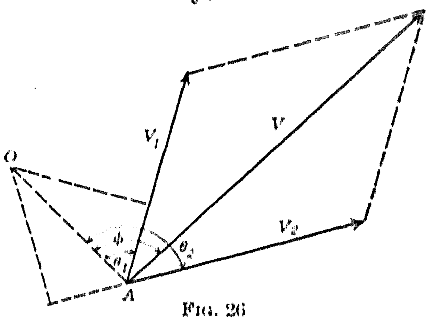


FIG. 26

This proof may obviously be extended to any number of vectors by combining the moment of any two of them into a resultant moment, combining this resultant with the moment of the third vector, etc., thus leading finally to the fundamental theorem of moments, namely:

The sum of the moments of any number of vectors with respect to a given point is equal to the moment of their resultant with respect to this point.

25. Composition of Angular Velocities. — Consider two angular velocities, ω_1 and ω_2, represented in Fig. 27 by the vectors OA and OB, respectively, and let the planes MN and RS perpendicular to these vectors be the planes of their moments. Assuming the direction of the vectors as indicated in the figure, the angular velocity ω_1 causes a right-handed rotation about the axis OA, which moves

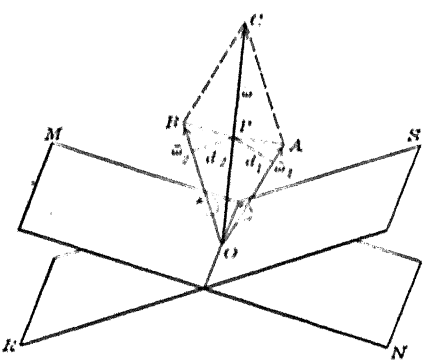


FIG. 27

any point P in the plane AOB outward towards the eye. Similarly the rotation ω_2 produces a rotation about the axis OB which

moves the point P backward away from the eye. The amount of the elevation or depression of the point P above or below the plane of the paper is proportional to the distance of P from either axis and the angular speed about this axis jointly. If, then, the perpendicular distances of P from OA and OB are denoted by d_1, d_2, respectively, the amounts of this elevation and depression will be equal provided that

$$OA \times d_1 = OB \times d_2.$$

But from the figure it is evident that this condition is equivalent to

$$\text{Area } OPA = \text{Area } OPB.$$

Therefore the condition that P shall remain stationary is that it shall lie on the diagonal of the parallelogram constructed on OA and OB as adjacent sides. This diagonal is therefore the axis of the resultant rotation.

Moreover, the angular displacement of every point in the plane AOB is proportional to the algebraic sum of the moments of $\bar{\omega}_1$ and $\bar{\omega}_2$ about this point. But by the theorem of moments proved in Art. 24, this is equal to the moment of OC about the given point. Hence OC represents, both in magnitude and position, the resultant of $\bar{\omega}_1$ and $\bar{\omega}_2$.

This theorem can evidently be extended to any number of angular velocities by combining them two at a time by means of the parallelogram law just obtained. The vector polygon construction explained in Art. 6 and 9 can therefore be extended to include angular velocities.

By similar reasoning it is also evident that angular accelerations may also be combined by means of a vector parallelogram or vector polygon.

26. Harmonic Motion. — In the motions so far considered the acceleration has been constant or zero. Next in simplicity are motions in which the acceleration is a linear function of the displacement. For example, in the case of the vibration of a spring, or of the string of a musical instrument, the acceleration is directly proportional to the displacement from the middle position, as explained further in Art. 105. The motion in this case is said to be **harmonic**. A harmonic motion is defined there-

fore as the motion of a particle which moves in a straight line with an acceleration directed towards the origin and proportional to its distance x from it.

The equation of motion in this case is therefore

$$a = -\omega^2 x,$$

where the constant of proportionality is denoted by ω^2 for reasons which will appear in the next article. Since $a = \frac{dv}{dt}$, this may be written

$$a = \frac{dv}{dt} = -\omega^2 x. \qquad (15)$$

To integrate this expression the variables may be separated by multiplying by dx. Then

$$\frac{dv}{dt}dx = -\omega^2 x dx,$$

or, since $\frac{dx}{dt} = v$, this becomes

$$v dv = -\omega^2 x dx,$$

which integrates into $$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c_1.$$

To determine the constant of integration c_1, let $v = 0$ when x attains a certain maximum value r. This is not an arbitrary assumption, but is characteristic of the motion, since in any motion of oscillation the point comes to rest at a certain distance from its initial position, and then reverses its direction of motion. Substituting $v = 0$, $x = r$ in the last equation, the constant is found to be $c_1 = \frac{\omega^2 r^2}{2}$, and the integral becomes $\frac{v^2}{2} = \frac{\omega^2 r^2}{2} - \frac{\omega^2 x^2}{2}$, which may be simplified into

$$v = \omega\sqrt{r^2 - x^2}. \qquad (16)$$

Since ω and r are constants, this relation expresses the speed v in terms of the displacement x. Evidently $v = 0$ when $x = r$, and $v = \omega r$ when $x = 0$, that is to say, the motion is an oscillation about the fixed point O, and the speed is greatest when passing through this point. The motion may be illustrated by means of a spiral spring to which a weight is attached. If pulled down

below the position of rest and then released, the weight will vibrate upward and downward, coming to rest at some distance r above and below its initial position, and attaining its greatest speed when passing through its initial position.

To obtain a second integral, replace v in Eq. (16) by its equal $\frac{dx}{dt}$. Then separating the variables, Eq. (16) becomes

$$\frac{dx}{\sqrt{r^2 - x^2}} = \omega dt,$$

which integrates into $\sin^{-1}\frac{x}{r} = \omega t + c_2$.

To determine the constant of integration c_2, let the angle whose sine is $\frac{x}{r}$ when $t = 0$ be denoted by ϵ. Then $c_2 = \epsilon$, and the integral becomes $\sin^{-1}\frac{x}{r} = \omega t + \epsilon$, or

$$x = r \sin(\omega t + \epsilon). \tag{17}$$

For the special case in which $\epsilon = 0$, this simplifies into $x = r \sin \omega t$.

To give a graphical illustration of the nature of harmonic motion, it is convenient to first put the equations of motion in the trigonometric form. Equation (17) is already in this form, namely,

$$x = r \sin(\omega t + \epsilon).$$

Similarly, Eq. (16) may be written

$$v = \frac{dx}{dt} = \omega r \cos(\omega t + \epsilon),$$

and Eq. (15) becomes

$$a = -\omega^2 x = -\omega^2 r \sin(\omega t + \epsilon).$$

The relations between the distance, speed, and acceleration as apparent in these equations are shown graphically in Fig. 28, which is drawn for ω equal to unity.

Equation (17) is characteristic of all motions of vibration, the sine being a periodic function repeating its value whenever the angle ωt changes by a multiple of 2π, or, what amounts to the same thing, whenever the time t changes by a multiple of $\frac{2\pi}{\omega}$.

Thus if $t = \frac{2\pi}{\omega}, 2 \cdot \frac{2\pi}{\omega}, 3 \cdot \frac{2\pi}{\omega}, \cdots, n \cdot \frac{2\pi}{\omega}$, the value of the sine is the same in each case, and consequently the displacement from the mean position is also the same. Therefore the moving point

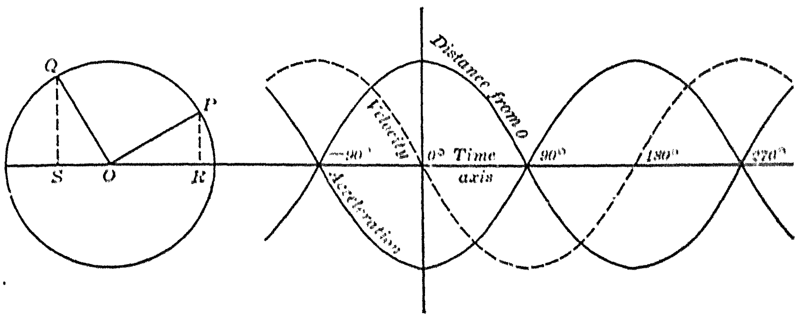


FIG. 28

occupies the same positions at intervals of time differing by $\frac{2\pi}{\omega}$.

The interval of time required for a complete oscillation is called the **period**, and will be denoted by P. Consequently $P = \frac{2\pi}{\omega}$, or, since $a = -\omega^2 x$, we have $\omega = \sqrt{\frac{a}{x}}$ in absolute value, and hence

$$P = 2\pi\sqrt{\frac{x}{a}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}. \tag{18}$$

The theory of vibration and its applications will be further considered in Chapter V.

PROBLEMS

55. A point moves with a harmonic motion such that its speed is 20 ft./sec. when it is 3 in. from the center of its motion, and 5 ft./sec. when 12 in. from the center. Determine the period and amplitude of the motion, and the maximum acceleration of the point.

56. A weight rests on the scale pan of a spring balance which is pulled down and then released, so that it executes a harmonic vibration having a period of 0.25 sec. Find the greatest possible amplitude of vibration so that the weight shall not leave the pan.

57. The spring of an automobile deflects 4 in. under a suddenly applied load. Find the period of the vibration.

27. Uniform Circular Motion. A special case of harmonic motion is that of the projection on a diameter of uniform circular motion. An example of such motion is furnished by a suspended weight swinging in a horizontal circle about a vertical axis. Such

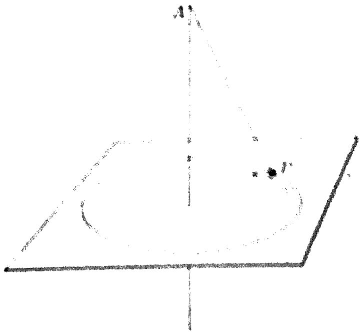


FIG. 29

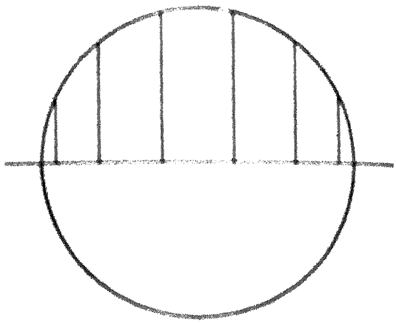


FIG. 30

an arrangement is called a conical pendulum (Fig. 29) and has a practical application in the Watt engine governor.

When viewed in the plane of the motion, the circular path is projected into a straight line along which the point P appears to move with varying speed, apparently stopping at either end and moving fastest at the center. The reason for this is that equal arcs are passed over in equal times, but since the projections of these arcs on a diameter decrease toward either end of the diameter, as shown in Fig. 30, the time of passing over a given distance along the diameter increases toward either end of the motion.

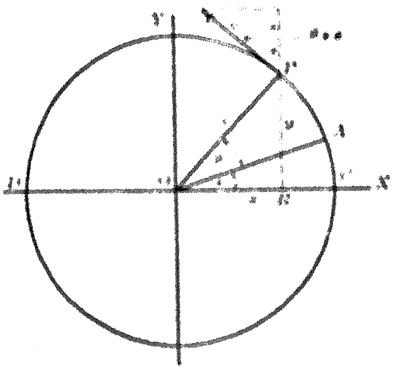


FIG. 31

To analyze the motion, consider a point P moving uniformly in a circle of radius r (Fig. 31). Draw any diameter DOC', and drop a perpendicular PR from P on this diameter. Let the point be at A when $t = 0$, and denote the angle AOC' by ϵ. Then if θ

denotes the angle described in the time t, we have from the figure

$$y = r \sin(\theta + \epsilon);$$

or, if ω denotes the angular speed with which θ is described, then $\theta = \omega t$, and hence

$$y = r \sin(\omega t + \epsilon).$$

This, however, is the same as Eq. (17) of Art. 26. Hence as P revolves uniformly in a circle, its diametral projection R executes a harmonic oscillation backward and forward along any diameter DOC.

From the figure the horizontal component of v is

$$v_x = v \sin(\theta + \epsilon),$$

or, since $\sin(\theta + \epsilon) = \frac{y}{r}$, and $v = r\omega$, this becomes $v_x = \omega y$. Moreover, since the equation of the path is $x^2 + y^2 = r^2$, we have $y = \sqrt{r^2 - x^2}$, and hence

$$v_x = \omega \sqrt{r^2 - x^2},$$

which is identical with Eq. (16) of the preceding article.

For uniform circular motion, the acceleration is directed toward the center and is of amount $\frac{v^2}{r}$ or $r\omega^2$, as shown in Art. 10 and 22. The horizontal component of this is $a_x = -\omega^2 r \cos(\theta + \epsilon)$, and since $\cos(\theta + \epsilon) = \frac{x}{r}$, this becomes

$$a_x = -\omega^2 x,$$

which agrees with the definition of harmonic motion given by Eq. (15) of Art. 26.

The time required for the point to make a complete revolution is called the **period**, and will be denoted by P, as in the preceding article. Since the angular displacement in one revolution is 2π radians, and the angular speed is ω rad./sec., we have

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x}{a_x}},$$

as in Eq. (18) of the preceding article.

The reciprocal of the period, namely, $n = \frac{1}{P}$, is called the **frequency**, and gives the number of oscillations per unit of time. The radius r is called the **amplitude** of the motion, and the angle

ϵ the **epoch** or **phase**. The latter is also called the **lead** when positive and the **lag** when negative. If the point starts from the initial line DOC, $\epsilon = 0$, and the equation of motion becomes simply $y = r \sin \omega t$.

It should be noted that harmonic oscillations have, in general, nothing whatever to do with motion in a circle. The illustration just given of the projection on a diameter of uniform circular motion is useful in giving a geometrical meaning to the quantities involved, and providing a simple means for remembering the various relations, but should never be regarded as a definition of harmonic motion.

28. Motions approximately Harmonic. — The occurrence of harmonic oscillations in mechanisms may be illustrated by the following simple examples.

Consider the motion of a simple pendulum, such as that of a bullet swinging in a small arc at the end of a long fine thread (Fig. 32). If the body swings freely, the only acceleration acting at P is that due to gravity, acting vertically downward. The effective acceleration along the path is then the tangential component of g, or $g \sin \alpha$, where α denotes the inclination of the radius OP to the vertical.

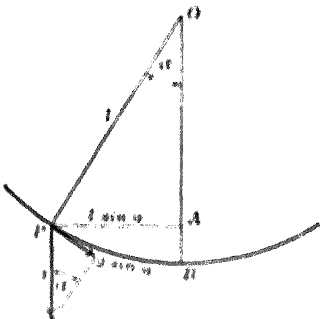


FIG. 32

Since g is constant and $\sin \alpha = \frac{x}{l}$, the effective acceleration is proportional to the distance x of P from the vertical OA. Hence if the path was the chord PA instead of the arc PB, the motion of P would be harmonic. For small arcs, or large radii, the sine, or chord, approximately coincides with the arc, and consequently small pendulum oscillations are approximately harmonic.

From Eq. (18) the period of a harmonic oscillation is

$$P = 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

In the present case the displacement is $l \sin \alpha$, and the acceleration along the path is $g \sin \alpha$. Substituting these values, the approxi-

mate value of the period, or time of swing, of a simple pendulum is found to be

$$P = 2\pi\sqrt{\frac{l}{g}}. \tag{19}$$

A more accurate expression for the period is given in Art. 112 and 113.

From this expression for the period an approximate value of g may be found experimentally by measuring the length l of a pendulum formed of a small heavy weight suspended by a long fine fiber, and observing its time of swing, or period P.

The crosshead of a steam engine affords another example of a motion which is approximately harmonic. In this case the connecting rod AB, Fig. 33, converts the uniform circular motion at B into a reciprocating motion at A. If the connecting rod was infinitely long, the motion of A would be harmonic. For connecting rods of ordinary length it is approximately harmonic. (See Art. 149.)

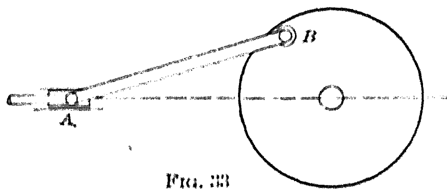


FIG. 33

An ingenious application of such motion has been made in Cornwall, Great Britain, in what is known as a "man-pump," used in raising and lowering miners to and from their work. The apparatus consists of a vertical rod or plunger, Fig. 34, which is given a reciprocating motion by means of a crank and connecting rod, as shown in the figure. On this rod are fastened at equal intervals a series of platforms, C_1, C_2, C_3, etc., while against the wall of the shaft are fastened at the same intervals another series of platforms D_1, D_2, D_3, etc. These are arranged at such distances apart that the travel of the plunger is just equal to the distance between two successive platforms. In operation each platform, for example C_1, is raised by the crank

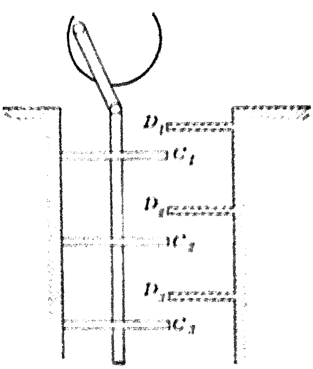


FIG. 34

and plunger until at its highest point it is level with the corresponding platform D_1. It then begins to descend, and is lowered until at its lowest position it is opposite the platform D_2. To descend the shaft, therefore, a man waits upon the platform D_1 until the platform C'_1 rises to that level. When C'_1 stops to reverse its motion, he steps upon it and is lowered to D_2. When the motion again stops, he steps off upon D_2 and waits for C'_2 to rise, in this way passing successively from one platform to another until he reaches the bottom. A continuous stream of men can thus be lowered and raised at the same time.

PROBLEMS

58. The crank of an engine is one foot long and makes 200 r. p. m. If the connecting rod is so long that the motion of the crosshead and piston may be regarded as harmonic, find their maximum speed and maximum acceleration.

59. In the preceding problem draw a curve showing the piston acceleration at all points of the stroke.

60. How many complete oscillations per minute will be made by a clock pendulum one meter long? (1 m. = 39.37 in.)

61. What is the proper length for a seconds pendulum at sea level in latitude 45°?

29. Relative Motion. — All motion is relative; that is to say, the nature of any motion depends on the point of view of the observer. For instance, if the motion of the bob of a conical pendulum is viewed from a point on the axis, either above or below the plane of the motion, the bob will appear to move uniformly in a circle, whereas if it is viewed edgewise in the plane of the motion, the bob will appear to have a linear harmonic motion, as explained in Art. 27. The motion of the north star, Polaris, is another example of this kind, for the rotation of the earth causes it to appear to an observer at the pole to move in a small circle around the pole, whereas to us in the northern hemisphere it seems to move east and west through equal small arcs on either side of the pole, with a motion similar to the harmonic vibration of a simple pendulum.*

* By observing Polaris with a transit at its greatest eastern and western elongations, and bisecting the angle between the lines so determined, the meridian of any place may be located.

Since in order to define a motion it is necessary to have some frame of reference to which it may be referred, the description of a motion depends on the particular coördinate system selected for reference. As there are no fixed lines or planes in nature, the description of any motion is therefore necessarily relative rather than absolute.

To illustrate, let A and B be any two points referred to an arbitrary coördinate system XOY, Fig. 35, and suppose that A receives a displacement AA', and B a displacement BB', relative to these axes. Then A has also received a displacement relative to B; that is to say, the distance and direction of A from B has been changed from AB to $A'B'$.

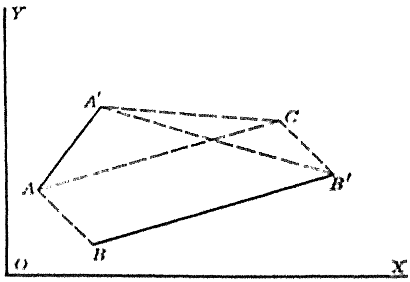


FIG. 35

The amount of this change of position of A relative to B may be found by combining the displacement of A relative to the axes with the displacement of B relative to the axes by means of a vector triangle. Thus draw AC parallel and equal to the displacement BB', and find the resultant $A'C$ of AA' and AC. Then $A'C$ represents the displacement of A relative to B. In other words, the original distance and direction of A from B was given by AB or CB', whereas the new distance and direction is $A'B'$. The change in displacement is therefore $A'C$, the closing side of the vector triangle $A'B'C$, which is obviously independent of the choice of coördinate axes.

The relative velocity of one point with respect to another may also be found by adding the vector velocities of each of these points with respect to a fixed set of coördinate axes. It is evident, as above, that the choice of a coördinate system is immaterial. In fact, this must necessarily be the case since the relative motion of one point with respect to another is independent of the choice of coördinate axes. The same proposition also holds for relative accelerations.

To further illustrate relative motion, consider the motion of a wheel rolling uniformly on a horizontal surface. Let ω denote

the angular velocity with which the wheel is revolving about an axis through its center. Then the linear speed of a point on the rim relative to the center C is $r\omega$ (Fig. 36). Since the wheel is assumed to roll without slipping, the center C has a linear velocity v relative to the ground. Therefore the velocity of a point P on the rim relative to the ground may be found by adding the velocity of P relative to C to the velocity of C relative to the ground. Thus, forming a vector parallelogram or triangle on $r\omega$ and v as sides, the resultant AP represents the velocity of P relative to the ground.

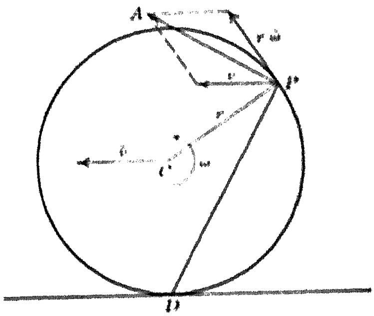


FIG. 36

For the highest and lowest points of the rim, v and $r\omega$ are parallel. Since they are also numerically equal, the speeds of these points relative to the ground are for the highest point $v + r\omega$, or $2v$, and for the lowest point $v - r\omega$, or zero. For this reason the upper portion of the rim of a moving wheel appears to an observer on the ground to move faster than the lower. This difference is often apparent in photographs of rapidly moving vehicles, the upper spokes in the wheels appearing blurred, whereas the lower are distinct.

An important application of relative motion occurs in considering the motion of a point which is referred to a set of moving axes. For instance, in considering the motion of a body on the surface of the earth, such as that of a gyroscope, it is necessary in certain cases to consider not only its motion relative to a set of axes fixed in the earth, but also the effect produced by the movement of these axes arising from the rotation of the earth and its orbital motion.

PROBLEMS

62. A boat is rowed at the rate of 5 mi./hr. across a river running at 3 mi./hr. In what direction must the boat be headed in order to move directly across the river, and with what speed will it then move?

63. Find the true course and speed of a vessel steering due north by compass at the rate of 10 knots through a 4-knot current setting southeast. Also

determine the alteration in direction by compass in order for the vessel to make a true northerly course. (1 knot = 6080 ft./hr.)

64. An astronomer in observing a star finds that it is necessary to correct his observation for the motion of the earth around the sun. Assuming that the speed of light is 984,000,000 ft./sec. and the mean speed of the earth in its orbit is 98,400 ft./sec., find the amount of the required angular correction α (Fig. 37).

(This is called the aberration of light, and was discovered in 1727 by James Bradley, afterward Astronomer Royal of England.)

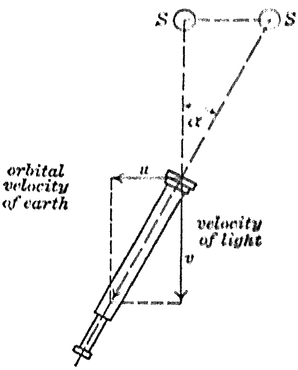


FIG. 37

65. The hour and minute hands of a clock are 6 in. and 9 in. long, respectively. Find the relative velocity of their extremities at 2 o'clock.

66. Two trains on parallel tracks, moving in opposite directions, pass in 5 sec. If the trains are each 350 ft. long and moving with the same speed, find this speed.

67. An aëroplane travels eastward along the equator at the rate of 100 mi./hr. If the circumference of the equator is 24,900 mi., how much shorter will the day be for the aviator?

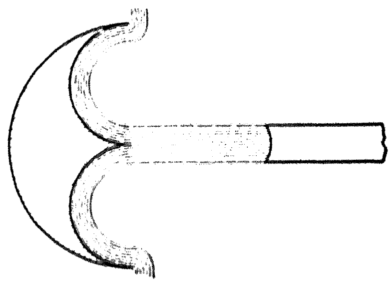


FIG. 38

68. A jet of water strikes a vane of a Pelton water wheel moving with half the speed of the jet (Fig. 38). If the cups forming the vane are hemispherical, so as to completely reverse the direction of the jet, show that the absolute velocity of the jet when it leaves the vane is zero.

SOLUTION. Let v denote the velocity of the jet and $\frac{v}{2}$ the velocity of the vane. Then when the jet impinges on the vane, its velocity relative to the vane is $v - \frac{v}{2}$ or $\frac{v}{2}$. When the jet leaves the vane, its velocity is reversed in direction, and consequently the absolute velocity of the jet on leaving is $\frac{v}{2} - \frac{v}{2} = 0$.

(Since there is no kinetic energy left in the water, the theoretical efficiency of the wheel under the conditions stated is unity. For the actual efficiency, see Prob. 109, Art. 41.)

30. Rotation. — Any displacement of a plane figure in its own plane may be regarded as a rotation about a definite axis perpen-

dicular to this plane. To prove this theorem, it is only necessary to consider the motion of a line joining any two points of the figure, since one line is sufficient to determine the position of a plane figure.

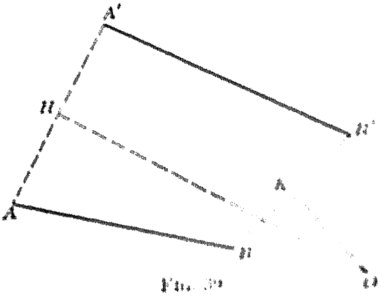


Fig. 39

Suppose, then, that a given straight line AB is displaced into the position $A'B'$ (Fig. 39). Then A could have reached A' by rotation about any point in the perpendicular OH erected at the middle of AA'. Similarly B could have reached B' by rotation about any point in the perpendicular OK. Hence the given displacement is equivalent to a rotation about O, the intersection of the perpendiculars OH and OK, which proves the theorem.

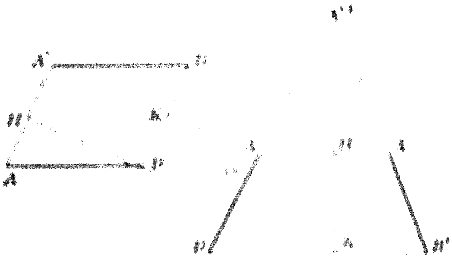


Fig. 40

If the perpendiculars OH and OK are parallel, the center of rotation O is infinitely distant. In this case AA' and BB' are equal and parallel, and hence the motion is a translation. Therefore a translation is equivalent to a rotation about an infinitely distant center.

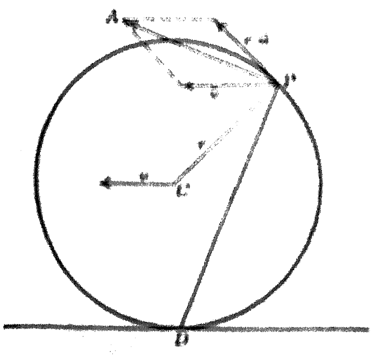


Fig. 41

If the perpendiculars OH and OK coincide, their intersection is apparently indeterminate. In this case, however, it is either at infinity, provided the motion is a simple translation, or it may be found by prolonging AB and $A'B'$ until they intersect, as shown in Fig. 40.

For the rolling wheel in the preceding article, the motion was there considered as a combined ro-

tation and translation. The motion of the point P on the rim is then equivalent to a single rotation about the point of contact D of the wheel with the ground (Fig. 41). That is, the resultant AP, obtained by adding the rotation $r\omega$ about C to its linear velocity v, is perpendicular to the radius DP, and consequently the motion of P is equivalent to a rotation about D with velocity AP.

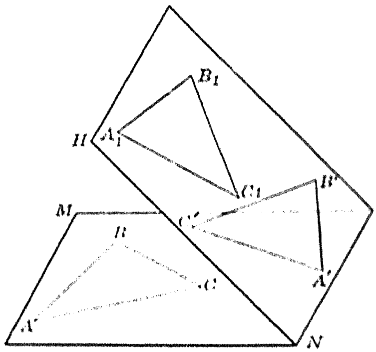


FIG. 42

The theorem just proved may be extended to motion in three dimensions. For this purpose it is sufficient to consider the motion of any three points forming a triangle, since three points not in the same straight line determine the position of a body in space.

Suppose then that the three given points forming the triangle ABC are displaced into some other position $A'B'C''$ (Fig. 42). Prolong the planes of the two figures until they intersect in a line NK. Then by revolving the first plane ABC about NK as an axis it can be made to coincide with the plane KH, and the triangle ABC will take up some position in this plane, say $A_1B_1C_1$. The problem is thus reduced to the case of planar motion, which by the theorem just proved is always equivalent to a single rotation. Hence any displacement whatever of a rigid body is always equivalent to one or more rotations about definite axes.

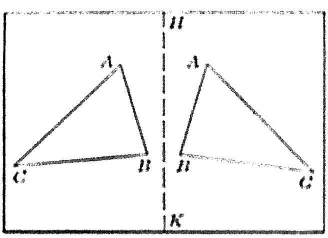


FIG. 43

A case of special interest is that which is met in plane geometry when two similar and equal triangles lie in the same plane but in reversed positions, as shown in Fig. 43. In this case there is no planar motion which can bring them into coincidence, and hence Euclid's method of superposition is frequently criticized. If, however, one of these triangles is revolved 180°

about an axis in its plane, as by folding the paper over upon itself along some line HK, it will return into the same plane but in a reversed position, and therefore the two triangles can now be made to coincide by a planar motion.

31. Instantaneous Center and Centrode. Any finite rotation may, in general, be broken up into a series of infinitesimal rotatives, each of which takes place about some definite point, which, however, changes as the motion proceeds. The point about which one of these infinitesimal rotations takes place is called the **instantaneous center**, since it is only momentarily the center of rotation.

For example, in the case of the rolling wheel considered above, all points of the wheel are rotating at any given instant about the point of contact D of the wheel with the ground. As the motion continues, this point of contact advances along the ground in the direction in which the wheel is moving. The motion of the wheel may therefore be considered as a succession of infinitesimal rotations about the point D which is itself in motion.

To illustrate further, suppose that a line AB moves continuously. Its ends, A and B, will then trace certain paths, as indicated in Fig. 44. Since the motion of A at any instant is in the direction of the tangent to its path, it is equivalent to a rotation about some point of the normal AO. Similarly the motion of B is equivalent to a rotation about some point in the normal OB. Hence the motion of the line AB at any given instant is equivalent to a rotation about O, the intersection of these normals. This point O is the instantaneous center. As the motion of AB proceeds, the intersection O of the normals changes continuously, and the motion therefore consists, as stated above, of a succession of infinitesimal rotations about a constantly moving, or instantaneous, center. The locus traced by the instantaneous center in its motion is called the **centrode**.

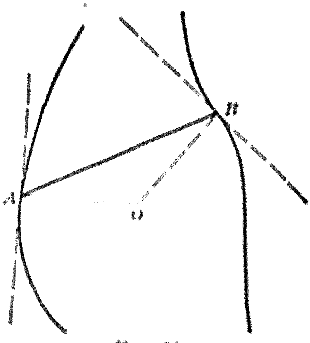


FIG. 44

Thus, if any line AB of a figure occupies successively the positions A_1B_1, A_2B_2, etc. (Fig. 45), and in each position lines are

drawn normal to the directions in which the ends A and B are moving, their intersections O, O_1, O_2, etc., will determine the corresponding instantaneous centers. If the motion of AB is continuous, the locus of O will also be continuous, the curve traced out being the centrode for the motion in question

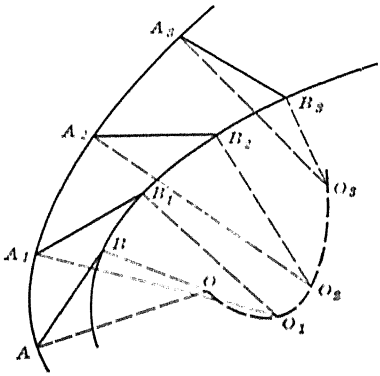


FIG. 45

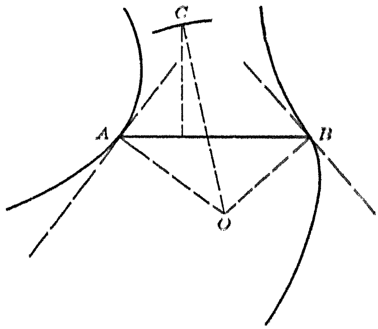


FIG. 46

When the centrode has been determined, the path of any point of the body can be found. For since at any instant the entire body is revolving about the instantaneous center O, any point of the body, such as C' (Fig. 46), is moving in an arc of a circle with OC' as radius. If, then, the location of O is determined for various positions of AB, and for each position a small arc is described with OC' as radius, the succession of these arcs (*i.e.* their envelope) will constitute the path of C'.

PROBLEM

69. Construct the centrode for the connecting rod BD of the **walking-beam** engine sketched in Fig. 47.

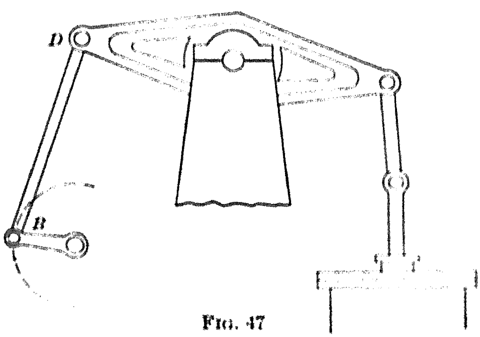


FIG. 47

32. Piston Speed. — As an application of the centrode, consider the motion of the connecting rod of an engine (Fig. 48). As the crank pin F' describes a circle of radius equal to the length

r of the crank, the crosshead F oscillates backward and forward along AG with an amplitude equal to the stroke of the piston, or $2r$. Since the path of F' is a circle, a normal to the path has the direction of the radius PF'. The path of F, however, is a straight line, and hence its normal is perpendicular to AG at F. The intersection O of these two normals is then the instantaneous center for the position FF'.

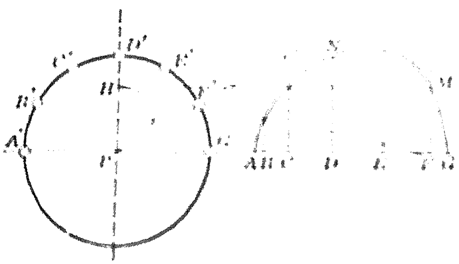


Fig. 48

As O is the instantaneous center for the position FF', the angular velocities of all points of FF' about O are the same at the given instant, and hence the linear speeds of F and F' are proportional to the radii OF and OF'. Hence if H denotes the intersection of the line of the connecting rod FF' with the vertical through P, then from similar triangles

$$\frac{\text{Speed of } F}{\text{Speed of } F'} = \frac{OF}{OF'} = \frac{PH}{PF'}.$$

If the crank PF' is revolving uniformly, the speed of F' is constant. In this case let PF' be drawn to such a scale as to represent the constant speed of F'. Then the intercept PH of the connecting rod on the vertical through P represents the speed of the piston for the given position of the connecting rod FF'. Hence by projecting the point H upon the ordinate through F, a point M is determined on the piston speed curve $GMNA$ (Fig. 48).

PROBLEMS

70. Derive the equation of the piston speed curve.

Solution. Let r denote the length of the crank PN, and l the length of the connecting rod FN (Fig. 49). Then if N with coördinates x, y is a point on the crank circle, and H with coördinates x, y' is the corresponding point on the piston speed curve, we have from the similar triangles FNT and NHS,

$$\sin\phi = \frac{NT}{l} = \frac{y}{l}, \text{ and } \tan\phi = \frac{HS}{SN} = \frac{HP - SP}{SN} = \frac{y' - y}{x}.$$

Since $\tan\phi = \frac{\sin\phi}{\sqrt{1-\sin^2\phi}}$, we also have

$$\frac{y'-y}{x} = \frac{\frac{y}{l}}{\sqrt{1-\frac{y^2}{l^2}}} = \frac{y}{\sqrt{l^2-y^2}}.$$

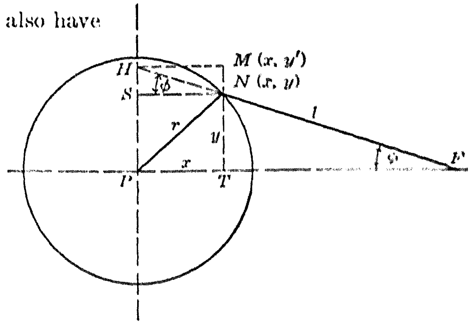


Fig. 49

Eliminating y by means of the relation $x^2+y^2=r^2$, this becomes

$$\frac{y'-\sqrt{r^2-x^2}}{x} = \frac{\sqrt{r^2-x^2}}{\sqrt{l^2-r^2+x^2}},$$

whence, after reduction, the equation of the piston speed curve finally becomes

$$y' = \sqrt{r^2-x^2} + x\frac{\sqrt{r^2-x^2}}{\sqrt{l^2-r^2+x^2}}.$$

For an infinitely long connecting rod, *i.e.* $l=\infty$, the second term of this equation disappears, and it simplifies into

$$y' = \sqrt{r^2-x^2},$$

which is the equation of a circle. Hence for a connecting rod of infinite length, the motion of the piston is harmonic.

71 Find the equation of the acceleration-time curve (Art. 16) for the piston, considering the connecting rod as infinitely long.

Solution. Since the ordinate y' to the speed curve represents the piston speed at any instant, the acceleration of the piston is $\frac{dy'}{dt}$. For a connecting rod of infinite length the speed curve is the circle

$$y' = \sqrt{r^2-x^2}.$$

Therefore $$a = \frac{dy'}{dt} = -x(r^2-x^2)^{-\frac{1}{2}}\frac{dx}{dt},$$

or, since

$$\frac{dx}{dt} = y' \text{ and } y' = \sqrt{r^2-x^2},$$

this simplifies into $$a = -x.$$

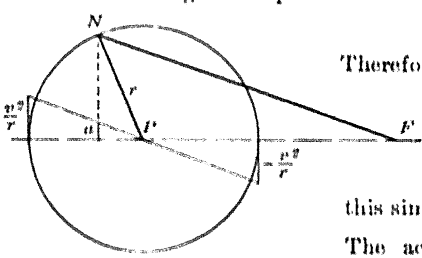


Fig. 50

The acceleration curve for a piston with infinitely long connecting rod is, therefore, a straight line, as shown in Fig. 50, the ordinate at either end being the same as the constant normal acceleration of the crank, namely, $\frac{v^2}{r}$.

This result will be applied in Chapter VII, under the balancing of engines.

72. In Prob. 69 draw the speed curve for D on the assumption that the angular speed of B is constant.

33. Axode. — In any displacement consisting of combined rotation and translation, the centrode was defined as the locus of the instantaneous center relative to axes fixed in space; that is to say, *outside the body*. During the displacement, however, the instantaneous axis about which the body is rotating may change its position in the body. In this case the instantaneous center also traces out a curve relative to axes fixed in the body and rotating with it. This curve is called the **axode**, since it determines the position of the instantaneous axis relative to the body.*

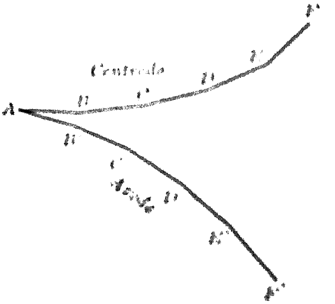


Fig. 51

Let A, B, C, D, etc., represent successive points of the centrode, that is, successive positions of the instantaneous center relative to axes outside the body, and let A', B', C', D', etc., represent successive points of the axode, that is, successive positions of this center in the moving figure (Fig. 51). Then the motion consists of a rotation about A until B' coincides with B, then about B until C' coincides with C, etc. That is to say, the motion is represented by the rolling of one polygon on the other. For continuous motion both polygons become continuous curves. Hence any displacement of a plane figure in its plane may be represented by the rolling of a plane curve attached to it on a plane curve fixed in space.

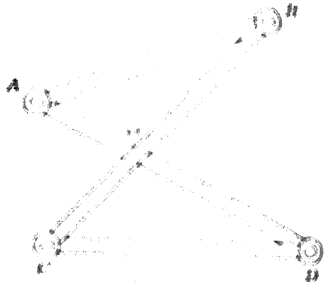


Fig. 52

This may be illustrated mechanically by four bars pivoted together at the ends, as shown in Fig. 52. Such an arrangement is called a **linkage**. Suppose that the link CD is fixed and the other links free to move. Then A rotates in

* The names centrode and axode are frequently used indiscriminately to denote the curve we have called the centrode. As there are two distinct curves traced out by the instantaneous center, it is convenient to have a special name for each, and the axode is therefore defined as above.

a circle about D as center, and B rotates about C as center. Hence AD and BC are normal to the paths of A and B, and therefore their intersection O is the instantaneous center about which the link AB is rotating. The locus of the instantaneous center, or centrode, so determined will be found to be an ellipse (Fig. 53). The axode in this case is also an ellipse, since it is simply the curve traced out by O when AB is fixed and CD revolves around it. The motion of AB with respect to CD may then be represented by the rolling of one of these ellipses upon the other.

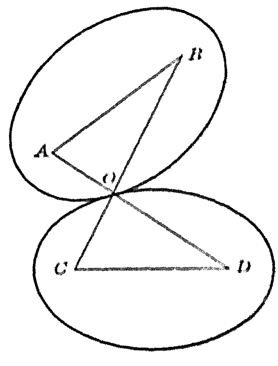


FIG. 53

The linkage shown in Fig. 52 is one frequently used in mechanisms for transmitting a varying velocity ratio.

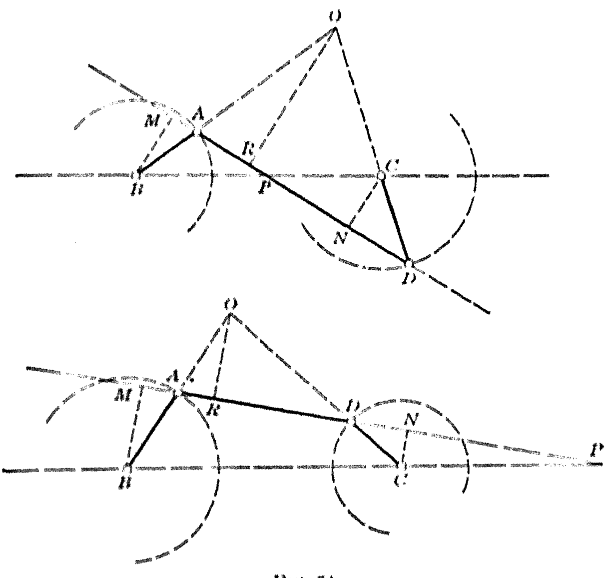


FIG. 54

34. Velocity Ratio in Mechanisms. — In the kinematics of machinery it is often necessary to determine the relative speed, or "velocity ratio," of the various moving parts. To illustrate,

let AB and CD represent two cranks connected by a rigid link AD (Fig. 54). Then from the speed of A at any instant let it be required to determine the speed of D at the same instant.

The instantaneous center for the link AD is found at the intersection of the normals to the paths of A and D. Since these paths are circles, the normals are directed along the radii BA and CD, intersecting in O, the required instantaneous center. At the instant considered, all points of AD are revolving about O with the same angular velocity. Hence the speed of any point of AD is proportional to its distance from O. In particular, the relative speeds of A and D are

$$\frac{V_A}{V_D} = \frac{OA}{OD}.$$

Now drop perpendiculars from B, C, and O upon the center line of the link AD. Then from the similar triangles BAM and RAO,

$$\frac{OA}{AB} = \frac{OR}{BM}.$$

Also from the similar triangles ORD and CND

$$\frac{OD}{CD} = \frac{OR}{CN}.$$

Let ω_A and ω_D denote the angular velocities of A and D. Then $\omega_A = \dfrac{V_A}{AB}$ and $\omega_D = \dfrac{V_D}{CD}$, and consequently

$$\frac{\omega_A}{\omega_D} = \frac{V_A}{V_D} \cdot \frac{CD}{AB} = \frac{OA}{OD} \cdot \frac{CD}{AB},$$

or, inserting the values of these ratios from the above, the speed ratio becomes

$$\frac{\omega_A}{\omega_D} = \frac{OR}{BM} \cdot \frac{CN}{OR} = \frac{CN}{BM}.$$

Therefore, *the angular speeds of the arms (or cranks) are inversely proportional to the perpendiculars from the fixed centers of motion upon the line of the link.*

Furthermore, since the triangles MBP and CNP are similar, $\frac{CN}{BM}=\frac{CP}{BP}$, and hence

$$\frac{\omega_A}{\omega_D}=\frac{CP}{BP}.$$

Consequently, *the angular speeds of the arms are inversely proportional to the segments into which the line of the link divides the line of centers.*

Motion is also frequently communicated by means of a flexible connector, such as a belt, rope, or chain drive (Fig. 55). In this case, if lines are drawn from the centers A and C to the points of contact B and D of the flexible connector, the motion is the same as though transmitted by a rigid linkwork $ABDC$. Hence the relative speeds of B and D are given by the rules just deduced.

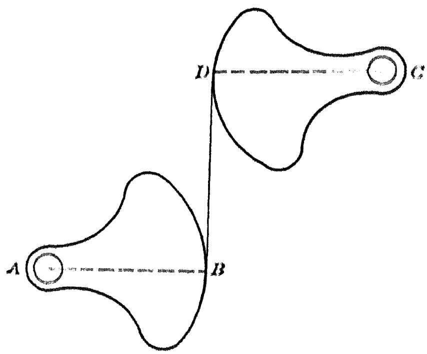


Fig. 55

The same is also true for motions produced by sliding contact, as in the case of cams or gears. Thus let Fig. 56 represent two cams turning about the fixed centers B and C, and let P be their point of contact. Draw through P the common normal RS and common tangent TQ. Then if circular arcs are drawn, having the same curvature as the cams at the point P, the motion of the cams for a small displacement will be the same as for these circular arcs. Let A and D be the centers of these arcs, both of which lie on the common normal RS. Then the motion at the instant considered is the same as for a rigid linkwork $BADC$. Consequently the above rules for determining the velocity ratio also apply to

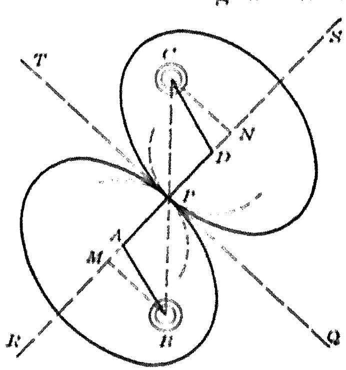


Fig. 56

this case. Thus drawing the line of centers BC, the velocity ratio of P about B and C is given by

$$\frac{\omega_c}{\omega_B} = \frac{BM}{CN} = \frac{BP}{CP}.$$

In many cases it is desirable to maintain a constant velocity ratio. The condition for this is that $\frac{BP}{CP}$ = constant, or since $BP + CP = BC$, which is also constant, the point P must remain fixed. Hence the condition for a constant velocity ratio is that the curves in contact must be such that their common normal at the point of contact shall always cut the line of centers in the same point. In designing gear wheels the curves which are usually chosen as having this property are epicycloid and hypocycloid.

For rolling contact the condition for constant velocity ratio is simply that the point of contact must always lie in the line of centers.

PROBLEMS

73. A set of gears meshed together as shown in Fig. 57 is called an epicyclic train. Find the velocity ratio between the first and last wheels.

SOLUTION. The contact surfaces of two toothed wheels which mesh with one another is called their *pitch surfaces*; that is to say, the pitch surfaces are the surfaces of two imaginary friction wheels which have the same axes as the toothed wheels and would also have the same angular velocities as the gears if one were to drive the other by rolling contact. A section of a pitch surface by a plane at right angles to its axis is called a *pitch line*. When the velocity ratio is constant, this pitch line becomes a *pitch circle* (Fig. 57).

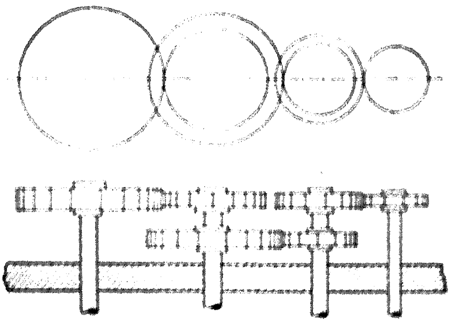


FIG. 57

The distance from a point on one tooth to a corresponding point on the next, measured along the pitch circle, is called the *circular pitch*. The number of these divisions is the same as the number of teeth, and since a fractional tooth is impossible, the circular pitch must be an aliquot part of the circumference of the pitch

circle. The number of teeth is also proportional to the diameter of the pitch circle, and the quotient of this diameter by the number of teeth is more convenient to use in practice than the circular pitch. In America this ratio is inverted and its reciprocal is called the *diametral pitch;* that is, diametrical pitch is defined as the number of teeth per inch of diameter of the pitch circle. For an epicyclic train the diametral pitch is evidently the same for each wheel.

In the present case the diameters of the pitch circles are assumed to be in the ratio 11 : 10 : 8 : 7 : 6 : 5. Therefore assuming a diametral pitch of 5 for the train, the number of teeth in the various wheels taken in the order of their size are, respectively, 55, 50, 40, 35, 30, 25. The velocity ratio between the first and last wheels, or *value of the train*, as it is called, is then given by

$$\text{Value of train} = \frac{\text{number of revolutions of last wheel}}{\text{corresponding number of revolutions of first wheel}}$$

$$= \frac{\text{continued product of number of teeth in drivers}}{\text{continued product of number of teeth in followers}}.$$

74. A set of friction bevel gears are arranged as shown in Fig. 58. Find the relative speeds of shafts A and D, the diameters of the pitch circles of the various wheels being $A = 3\frac{1}{2}$ in., $B = 5$ in., $C = 8$ in., $D = 6$ in.

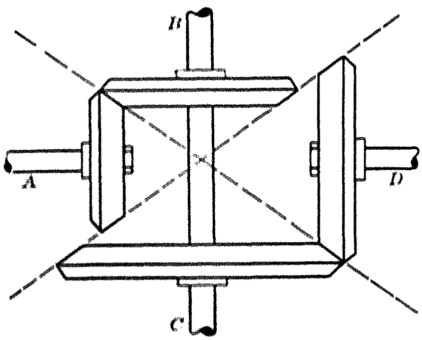


FIG. 58

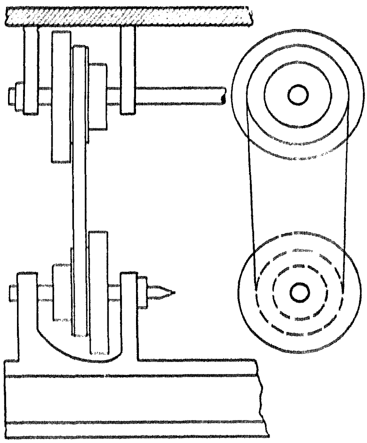


FIG. 59

75. A lathe is run from a countershaft by two sets of three-stepped pulleys, as shown in Fig. 59. If the pulleys are 6 in., 10 in., and 14 in. in diameter, respectively, find the velocity ratios obtainable.

76. Show that if a belt is crossed, its length is constant, provided the sum of the radii of corresponding pulleys is constant, as in the preceding problem.

SOLUTION. From Fig. 60, the length of the belt is

$$l = 2(\text{arc } AB + BC + \text{arc } CD)$$

$$= 2\left[\left(\frac{\pi}{2} + \theta\right)r_1 + \sqrt{d^2 - (r_1 + r_2)^2} + \left(\frac{\pi}{2} + \theta\right)r_2\right], \text{ where}$$

d is the distance between centers; or, since $\sin\theta = \frac{r_1 + r_2}{d}$, this may be written

$$l = 2\left[\left(r_1 + r_2\right)\left(\frac{\pi}{2} + \sin^{-1}\frac{r_1 + r_2}{d}\right) + \sqrt{d^2 - (r_1 + r_2)^2}\right],$$

which is constant, provided $r_1 + r_2$ is constant.

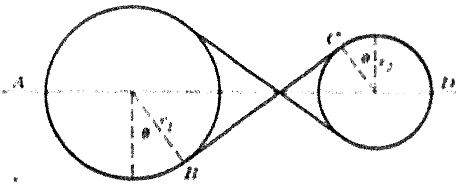


FIG. 60

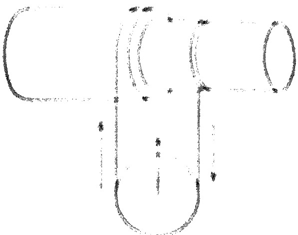


FIG. 61

77. In the differential pulley shown in Fig. 61 the rope is unwound from the small drum and wound up on the large, or *vice versa*. If the radii of the drums are R and r, respectively, find the speed with which the weight is raised or lowered when the drums revolve at one revolution per second.

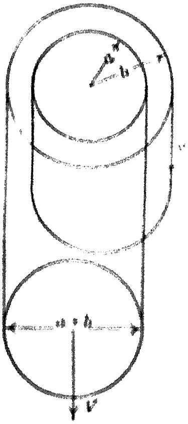


FIG. 62

78. The upper block of a Weston differential pulley consists of two sheaves of radii a and b, fastened rigidly together, and a lower block consisting of a single sheave of diameter $a + b$. An endless chain passes around all three sheaves, as indicated in Fig. 62. Show that the velocity ratio between the driving chain and the weight raised is $\frac{v}{V} = \frac{2b}{b - a}$, and that consequently the ratio between the driving force F and the weight W raised is $\frac{F}{W} = \frac{b - a}{2b}$.

79. If the diameters of the pulleys in the upper block of a Weston pulley are 7 in. and 8 in., find the theoretical advantage of the mechanism, neglecting friction.

80. In Prob. 79, if the weight is to be raised at the rate of ½ ft./sec., find the rate at which the hoisting cable must move.

81. In planers and shapers the cut is made in one direction, the return stroke being idle. To make the return stroke as quickly as possible, one of the means commonly used is the Whitworth quick return motion, shown in

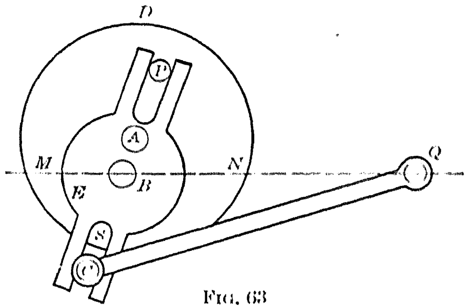


FIG. 63

Fig. 63. In the figure A is the fixed center of rotation of the crank AP, and B is the fixed pivot about which the arm BC rotates. The length of the stroke is evidently $2\ BC$, and the lengths of time taken for the forward and back strokes are proportional to the arcs MDN and NSM. The ends of the stroke occur when P is at M and N.

Design a Whitworth quick return motion to have periods as 2 to 1, and stroke of tool = 4 in.

35. Sectorial Velocity. — Let $\bar{v}_1$, $\bar{v}_2$ represent two successive vector velocities of a moving point. Then the third side of the triangle formed on $\bar{v}_1$ and $\bar{v}_2$ represents the change or vector acceleration, $\bar{a}$ (Fig. 64). By the theorem of moments, the sum of the moments of any number of vectors about a given point is equal to the moment of their resultant about this point. Hence, in the present case, the sum of the moments of $\bar{v}_1$ and $\bar{a}$ about any point O is equal to the moment of $\bar{v}_2$ about this point. Suppose now that O is chosen on the line along which $\bar{a}$ acts (Fig. 64). Then the moment of $\bar{a}$ about this point is zero, and therefore the moments of $\bar{v}_1$ and $\bar{v}_2$ about O are equal. Evidently the converse of this theorem is also true; namely, if the moments of $\bar{v}_1$ and $\bar{v}_2$ about any point O are equal, the moment of the acceleration $\bar{a}$ about O is zero. Hence:

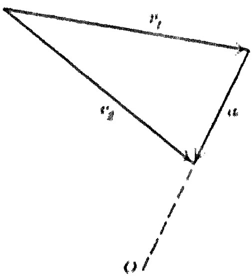


FIG. 64

If the moments of two successive vector velocities about any given point are equal, the vector acceleration passes through this point.

This theorem leads to an important special case of motion; namely, that in which the moment of the velocity of a moving point with respect to a fixed center O is constant. That is, if $\bar{v}$ represents the velocity at any instant, and r its perpendicular distance from a fixed center O, then the motion is such that the product $\bar{v}r$ is constant, say

$$vr = K. \qquad (20)$$

Since the moment K is constant, it may be represented by a fixed vector (Fig. 65). Therefore, the motion is planar, since the velocity v must always lie in the plane MN perpendicular to K.

The moment of the velocity v about O is also represented geometrically by twice the area of the triangle OAB. Therefore, if

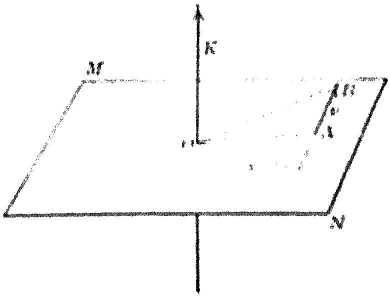


FIG. 65

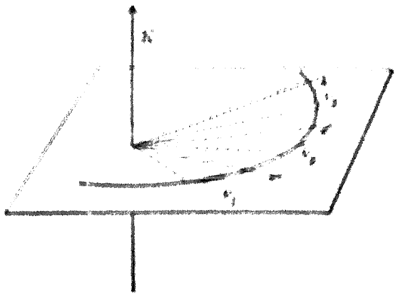


FIG. 66

the velocities of the moving point at different instants are represented by vectors tangent to the path, the areas of the triangles having O for a common vertex and these vector velocities as bases are all equal, since each is equal to the same constant K (Fig. 66). Now multiplying both sides of Eq. (20) by dt, it becomes

$$rvdt = Kdt,$$

or, since $vdt = ds$, where ds is the length of the path described in the time dt, this may be written

$$rds = Kdt.$$

Hence, if ds_1, ds_2, ds_3, etc., represent displacements occurring in equal intervals of time dt, and r_1, r_2, r_3, etc., represent the perpendicular distances of each of these displacements from O, respectively, then

$$r_1ds_1 = r_2ds_2 = r_3ds_3 = \cdots = Kdt,$$

since each is equal to Kdt, where K is constant and dt is assumed to be the same for all. Consequently the areas of the triangles formed on each displacement ds as base and having O for a common vertex are equal (Fig. 67). That is to say, the areas swept over by the radius vector in equal intervals of time are equal.

The area swept over in a unit of time, say one second, is called the **sectorial velocity.** The preceding demonstration may then be summed up by saying that when a point so moves that its vector acceleration constantly passes through a fixed center, its motion lies in one plane, and its sectorial velocity is constant. More briefly: —

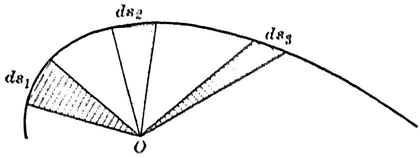


FIG. 67

Under central acceleration the sectorial velocity is constant.

The converse of this theorem is obviously true also, namely: —

When the sectorial velocity is constant, the acceleration is directed toward a fixed center.

36. Kepler's Laws of Planetary Motion. — From a long series of observations of the planets, especially of Mars, made by the Danish astronomer Tycho Brahe, in his observatory near Prague, his assistant, Johannes Kepler (born 1571, died 1630), deduced three laws which completely describe planetary motion. These laws are:

I. *The radius drawn from a planet to the sun describes equal areas in equal times* (i.e. *its sectorial velocity is constant*).

II. *Every planet describes an ellipse having the sun at a focus.*

III. *For different planets the squares of the times of describing their orbits are proportional to the cubes of their major axes.*

The statement of these laws marked an important epoch in the development of mechanics, for although purely kinematical, they led Newton (born 1642, died 1727) to the discovery of the law of attraction (or gravitation), and thus laid the foundation of modern mechanics.

The conclusions obtained by Newton from Kepler's three laws will now be derived.*

From the first law it is evident from the results deduced in the preceding article that the orbits of the planets are plane curves,

* See W. W. R. Ball: "Essay on Newton's *Principia.*"

Percival Frost: Newton's *Principia*, Sec. I, II, III. The method of analysis used here is of course different from that followed by Newton as given in these references.

and also that they move under a central acceleration directed toward the sun. This led Newton to the idea of an attractive force which caused the planets to continually "fall toward the sun"; that is, to move around the sun in an elliptical orbit instead of receding from it.

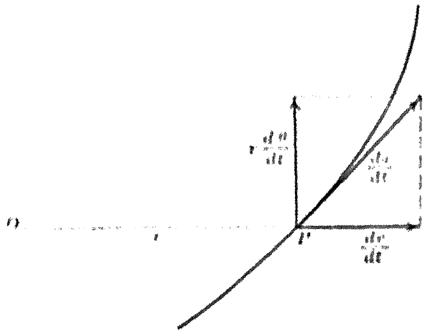


FIG. 68

From the second law the relation between the distance of a planet from the sun and its central acceleration may be found. For this purpose let the angular velocity of the planet about the sun be denoted by $\omega = \frac{d\theta}{dt}$, and resolve the tangential velocity $\frac{ds}{dt}$ into two components, one radial, of amount $\frac{dr}{dt}$, and the other perpendicular to the radius, of amount $r\omega$, or $r\frac{d\theta}{dt}$ (Fig. 68). Then from the vector triangle,

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \qquad (21)$$

The sectorial velocity is the rate of increase of any sector S, or MOP, as represented in Fig. 69. Hence if dS, or NOP, represents the increment to this sector in the time dt, the sectorial velocity is $\frac{dS}{dt}$. From the trigonometric expression for the area of a triangle, we have

FIG. 69

$$dS = \tfrac{1}{2} r\,ds \sin\alpha,$$

where α represents the angle between the radius OP and a tangent to the path at P. Since $ds \sin\alpha = r\,d\theta$, this becomes

$$dS = \tfrac{1}{2} r^2 d\theta, \qquad (22)$$

and hence

$$\frac{dS}{dt} = \tfrac{1}{2} r^2 \frac{d\theta}{dt}.$$

From Kepler's first law the sectorial velocity is constant; that is, $\frac{dS}{dt}=K$. Hence the above equation becomes $\frac{d\theta}{dt}=\frac{2\,K}{r^2}$, and substituting this value of $\frac{d\theta}{dt}$ in Eq. (21), the latter becomes

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2+\frac{4\,K^2}{r^2}.$$

To find the acceleration, this expression may be differentiated with respect to the time. Then

$$2\,\frac{ds}{dt}\,\frac{d^2s}{dt^2}=2\,\frac{dr}{dt}\,\frac{d^2r}{dt^2}-\frac{8\,K^2}{r^3}\,\frac{dr}{dt}. \tag{23}$$

Here $\frac{d^2s}{dt^2}$ is the tangential component of the radial acceleration a_r. If then α denotes the angle between them (Fig. 70),

$$\frac{d^2s}{dt^2}=a_r\cos\alpha,$$

or, since $\cos\alpha=\frac{dr}{ds}$, this becomes

$$\frac{d^2s}{dt^2}=a_r\,\frac{dr}{ds}.$$

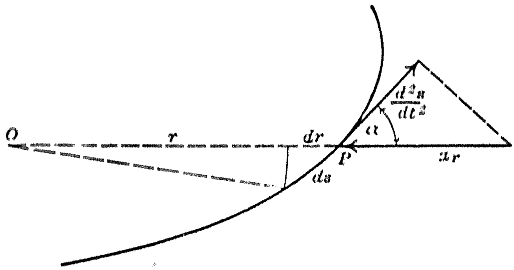


FIG. 70

Multiplying both sides of this expression by $\frac{ds}{dt}$, it becomes

$$\frac{d^2s}{dt^2}\,\frac{ds}{dt}=a_r\,\frac{dr}{dt},$$

and substituting this result in Eq. (23), we have

$$a_r\,\frac{dr}{dt}=\frac{d^2r}{dt^2}\cdot\frac{dr}{dt}-\frac{4\,K^2}{r^3}\,\frac{dr}{dt},$$

whence

$$a_r=\frac{d^2r}{dt^2}-\frac{4\,K^2}{r^3}. \tag{24}$$

To further evaluate this expression for the radial acceleration it is necessary to know the relation between r and t; that is to say,

the equation of the path. The second law, however, states that this is an ellipse, the equation of which in polar coordinates is

$$r = \frac{l}{1 + e\cos\theta},$$

where e denotes the eccentricity and l the semi-latus-rectum. Now by changing the variable we have

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = \frac{dr}{d\theta}\,\frac{2\,K}{r^2} = -2\,K\,\frac{d}{d\theta}\left(\frac{1}{r}\right),$$

and also $$\frac{d^2r}{dt^2} = \frac{d}{dt}\left[-2\,K\,\frac{d}{d\theta}\left(\frac{1}{r}\right)\right] = \frac{d}{d\theta}\left[-2\,K\,\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]\frac{d\theta}{dt}$$

$$= -2\,K\,\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\frac{d\theta}{dt} = -\frac{4\,K^2}{r^2}\,\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right). \tag{25}$$

Then, since $\dfrac{1}{r} = \dfrac{1 + e\cos\theta}{l}$ we have

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{-e\sin\theta}{l} \quad\text{and}\quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{-e\cos\theta}{l}.$$

But from the equation of the path $\dfrac{-e\cos\theta}{l} = \dfrac{1}{l} - \dfrac{1}{r}.$

Hence $\dfrac{d^2}{d\theta^2}\left(\dfrac{1}{r}\right) = \dfrac{1}{l} - \dfrac{1}{r}$, and inserting this value in Eq. (25), the latter becomes

$$\frac{d^2r}{dt^2} = -\frac{4\,K^2}{r^2}\left(\frac{1}{l} - \frac{1}{r}\right).$$

Substituting this value of $\dfrac{d^2r}{dt^2}$ in Eq. (24), we have then finally

$$a_r = -\frac{4\,K^2}{r^2}\left(\frac{1}{l} - \frac{1}{r}\right) - \frac{4\,K^2}{r^3},$$

which simplifies into $$a_r = -\frac{4\,K^2}{l\,r^2}. \tag{26}$$

Since this expression is negative and K and l are constants, the conclusion which may be drawn from Kepler's first and second laws is that

The central acceleration of a planet varies inversely as the square of its distance from the sun and is directed toward the sun.

From this Newton concluded that there is a force of attraction between a planet and the sun which also varies inversely as the square of the distance between them. This is called the Newtonian law of attraction, or gravitation. It is also sometimes called the law of inverse squares.

From Kepler's third law the value of the constant K may be found, and thus the numerical value of the radial acceleration a_r be determined.

Since the orbit is an ellipse, the area swept over in a complete revolution is πab. Since $\frac{dS}{dt} = K$, we have $dS = Kdt$ and also

$$\int dS = \int Kdt.$$

If, then, T denotes the time in which the orbit is described, since $\int dS = \pi ab$, and $\int Kdt = KT$,

$$\pi ab = KT,$$

whence

$$K = \frac{\pi ab}{T}.$$

Also, from analytic geometry the value of the latus rectum of an ellipse in terms of its semiaxes is

$$l = \frac{b^2}{a}.$$

Inserting these values of l and K in Eq. (26), it becomes

$$a_r = \frac{-4\left(\frac{\pi ab}{T}\right)^2}{\frac{b^2}{a} r^2} = -\frac{4\pi^2 a^3}{r^2 T^2}.$$

By the third law, $\frac{a^3}{T^2}$ is constant for all the planets. Consequently the factor by which the inverse square of the distance is multi-

plied is the same for all the planets, and therefore depends only on the sun.

The greatest of Newton's achievements consisted in perceiving that natural law was universal in its scope, and that the same law must apply to bodies on the surface of the earth as to the solar system. To verify his ideas Newton first applied them to the moon, as being nearest to the earth and its motion more precisely known. Assuming that the moon's orbit is circular, which is approximately correct, $a = r$, and the acceleration of the moon toward the earth is

$$a_{moon} = -\frac{4\pi^2 r}{T^2}.$$

From the law of inverse squares, the acceleration of a body at the surface of the earth must be inversely proportional to the square of the radius of the earth, say R. Consequently

$$\frac{a_{earth}}{a_{moon}} = \frac{r^2}{R^2},$$

and hence $$a_{earth} = a_{moon}\frac{r^2}{R^2} = -\frac{4\pi^2 r^3}{T^2 R^2}.$$

Assuming that the radius of the moon's orbit is sixty times the radius of the earth, and that the circumference of the earth is $4 \cdot 10^7$ meters, we have

$$r = 60\,R, \quad 2\pi R = 4 \cdot 10^7 \text{ m.}, \quad T = 27 \text{ da. } 7 \text{ hr. } 43 \text{ min.},$$

whence $$a_{earth} = 9.74 \text{ m./sec.}^2.$$

From actual measurements the acceleration of a body at the surface of the earth is found to be about

$$a_{earth} = 9.81 \text{ m./sec.}^2.$$

The discrepancy between these results arises from the approximate nature of the numerical values from which they were obtained. With more precise measurements they are found to agree closely.

CHAPTER II

FUNDAMENTAL DYNAMICAL PRINCIPLES

37. The Laws of Motion. — The conclusions which may be drawn from Kepler's laws of planetary motion, explained in Art. 36, led Newton (b. 1642, d. 1727) to the statement of three dynamical axioms, or universal laws, underlying the motion of all bodies in the universe. These laws were not original discoveries with Newton, as might be inferred, but summarized the work of all preceding investigators in mechanics. The first two laws were undoubtedly known to Galileo (b. 1564, d. 1642), and the second and third had been tacitly assumed by Huyghens (b. 1629, d. 1695), Wren (b. 1632, d. 1723), Hooke (b. 1635, d. 1703), and others, in various applications of mechanics. Kepler's laws, however, enabled Newton to obtain such a clear insight into the fundamental relations between force and motion that he was able to formulate these relations in simple axioms, thus laying the foundation for a scientific system of mechanics.

The three laws of motion as stated by Newton are as follows:

LAW I. — Every body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by impressed force to change this state.

LAW II. — Change of motion is proportional to the impressed force, and takes place along the straight line in which the force acts.

LAW III. — To every action there is always an equal and opposite reaction; or, in other words, the mutual actions of two bodies are always equal and oppositely directed.

38. The Fundamental Equation. — The three laws just stated may be expressed by a single simple equation which is the basis of the entire subject of dynamics. This fundamental equation is

embodied in Law II. Here "change of motion" is equivalent to "acceleration." Consequently this law may be expressed by saying that

The impressed (or external) force acting upon a body is proportional to the acceleration produced.

This means that the ratio of force to acceleration is constant for any given body. Denoting the external force by F, the acceleration produced by a, and their constant ratio by m, this law, expressed in symbolic form, is simply

$$\frac{F}{a} = m.$$

To obtain Newton's first law from this equation, write it in the form

$$F = ma.$$

Then if $F = 0$, we must also have $a = 0$, because m is a finite constant different from zero. Hence if no force acts, the acceleration is zero, and therefore the velocity is either zero or constant, as stated in Law I.

The third law simply expresses the equality of F and ma. F in this case is the "action," or tendency to produce motion, and ma is the "reaction," or inertia force set up. The third law may therefore be regarded as expressing the equality of cause and effect. The nature of the action between two bodies may be of any sort whatever, and may be transmitted by means of any number of intervening bodies. For instance, the mutual action of two heavenly bodies, or of two bodies connected by a rigid rod or an elastic spring, or attracting or repelling each other by magnetic or electric agencies, are all illustrations of the third law. In general, if m_1 and m_2 denote the masses of two bodies, and their respective accelerations are resolved into rectangular components, $\frac{d^2x}{dt^2}$, etc., Newton's third law is expressed by the relations

$$m_1\frac{d^2x_1}{dt^2} = -m_2\frac{d^2x_2}{dt^2}; \quad m_1\frac{d^2y_1}{dt^2} = -m_2\frac{d^2y_2}{dt^2}; \quad m_1\frac{d^2z_1}{dt^2} = -m_2\frac{d^2z_2}{dt^2}.$$

39. Mass. — It is found by experiment that the ratio of impressed force to acceleration produced depends on the nature of

the body on which the force acts. For instance, in the case of two bodies of exactly the same size and shape, but of different materials, say lead and wood, it is found that the same force F will produce a greater acceleration in the wooden block in a given time than in the lead. In other words, the lead block offers a greater resistance to change of motion than the wooden one; or, what amounts to the same thing, the ratio $\frac{F}{a}$, or m, is greater for the lead block than for the wooden one. Thus the ratio m, which is constant for any given body, is different for different bodies. This is expressed by saying that with each body there is associated a definite constant m (or ratio of force to acceleration), which expresses the amount of its resistance to change of motion, or its **inertia**, as it is called. This constant m, which measures the inertia of a body, is called its **mass**.

40. Relation between Mass, Weight, and Force. — The relation between mass, weight, and force is of such fundamental importance that several explanations of this relation are here given.

I. In every country a certain mass is chosen as a standard, and the mass of any other body determined by comparison with this standard.

The weight of this unit mass, that is, the pull of the earth on it due to the attraction of gravitation, is a force which may be conveniently chosen as the unit of force. This unit is convenient rather than scientific, as the weight of a body depends on its location with reference to the center of the earth. The extreme variation, however, is only about $\frac{1}{3}$ of 1 per cent, so that in engineering computations and in practical affairs the weight of one pound mass is commonly used as the unit of force.

Whatever units of force and mass are chosen, they must satisfy the relation $F = ma$. For this reason only one can be chosen arbitrarily, as when one is chosen the other is defined by this relation. Now the average acceleration g of a body falling freely in a vacuum is found to be about 32.2 ft./sec.^{2}, or 981 cm./sec.^{2}. That is to say, if the force acting on a certain mass is that due to its own weight, an acceleration of amount g is produced. Since force is proportional to acceleration, if this same force acts upon a mass g times as great, the acceleration produced will be unity.

Therefore if the weight of unit mass is chosen as the unit of force, and mg is assumed as the unit of mass, the fundamental relation becomes

$$W = mg \times 1;$$

that is, $W = mg$, or $m = \frac{W}{g}$. In all technical and practical computations, therefore, where the inertia of a body is measured by its weight, and forces are expressed in pounds (or grams) weight, the equation $F = ma$ must be used in the form

$$F = \frac{W}{g}a.$$

It is especially necessary to distinguish clearly between the two senses in which the English word "pound" is used, as no one thing in mechanics is so likely to cause mistakes as confusion on this point. To a physicist one pound invariably means one pound mass, whereas to an engineer, contractor, or tradesman it always means a pound weight, *i.e.* a force, and not mass at all.

In the French system the same confusion arises by the use of the same word "gram" to express two entirely separate ideas. To obviate this difficulty new words have sometimes been proposed, but so far have not been favorably received.

II. From the fundamental equation $F = ma$ it is evident that the masses of two bodies may be compared by measuring either (1) their relative accelerations under the action of equal forces, or (2) the relative forces required to produce equal accelerations in the bodies. The former method is used for scientific purposes, and is based on the definition of a unit of force. This is called the absolute unit of force, and is defined as the force which will produce unit acceleration in unit mass. In the English system the unit of mass is the pound and the absolute unit of force is the poundal. That is, one poundal is defined as the force which will give to a mass of one pound an acceleration of 1 ft./sec.^{2}. In the French system the unit of mass is the gram and the absolute unit of force is the dyne. That is, one dyne is the force which will give to a mass of one gram an acceleration of 1 cm./sec.^{2}.

In the second method the relative forces are chosen as the weights of the bodies, which produce the same constant accelera-

tion g in each case. Thus, for practical purposes, masses are compared by their weights.

Now from the fundamental equation $F = ma$, using the absolute unit of force, we have

$$\text{Force in poundals} = \text{mass in pounds} \times \text{accel. in ft./sec.}^2,$$

whereas, using the technical or gravitation unit of force (weight), we have

$$\text{Weight of one pound} = \text{mass of one pound} \times g \text{ ft./sec.}^2.$$

Hence,

$$\text{Weight of one pound} = g \text{ poundals}.$$

Therefore, in order that the fundamental equation $F = ma$ may apply to technical units, the force in poundals must be divided by g; that is to say,

$$\begin{aligned}\text{Force in pounds weight} &= \frac{\text{force in poundals}}{g} \\ &= \frac{\text{mass in pounds} \times \text{accel. in ft./sec.}^2}{g} \\ &= \frac{\text{mass in pounds}}{g} \times \text{accel. in ft./sec.}^2.\end{aligned}$$

The technical or gravitation unit of mass is therefore g times as great as the absolute unit of mass.

III. The momentum of a body is defined as the product of a constant m, called the mass of the body, by its velocity v.

Newton's second law may then be expressed by saying that the force acting on a body at any instant is proportional to the time rate of change of its momentum, or, in symbolic form,

$$\text{Force} = K\frac{d}{dt}(mv) = Km\frac{dv}{dt} = Kma,$$

where K denotes the constant factor of proportionality. The value of this constant K depends on the system of units assumed.

Absolute System. — For scientific purposes it is convenient to define the unit of force as that force which will produce unit acceleration in unit mass. With this definition, F, m, and a are all unity simultaneously, and hence the constant K is also unity.

Consequently, in scientific or absolute units,

$$\text{Force} = \text{mass} \times \text{acceleration}.$$

In the absolute system the unit force is called the poundal in the English system and the dyne in the French system.

Gravitation System. — For technical and practical purposes the unit force is defined as a force equal to the weight of a unit mass. Since the unit force so defined is weight, or attraction of gravitation, the acceleration a becomes that due to gravity g, and hence

$$F = Kmg.$$

By definition, however, F is unity when m is unity. Hence

$$1 = Kg, \text{ or } K = \frac{1}{g}.$$

Therefore, in gravitation units

$$\text{Force} = \frac{\text{mass} \times \text{acceleration}}{g}.$$

In the gravitation system, the unit force is called the pound in the English system and the gram in the French system.

PROBLEMS

82. A train weighing 100 T., excluding engine, runs up a 1 per cent grade with an acceleration of 1 ft./sec.^{2}. Assuming the frictional resistance to be 10 lb./T., find the pull on the draw bar of the engine.

83. A Pacific type passenger locomotive weighs 207,000 lb., of which 131,200 lb. is on the drivers. Weight of tender 113,000 lb. Maximum tractive effort 28,000 lb. Assuming frictional resistances to be 6 lb./T., find how heavy a train this engine can haul up a 1 per cent grade. Also find how long it will take to acquire a speed of 40 mi./hr. with this train on the level.

84. A 500 T. train acquires a speed from rest of 10 mi./hr. in 5 min. How long would it take a 750 T. train, drawn by the same engine, to acquire a speed of 12 mi./hr.?

85. A 3 T. elevator has an acceleration at starting of 5 ft./sec.^{2}. What is the greatest and least tension in the cable?

86. A train of six passenger coaches, each coach weighing 113,550 lb. and containing 88 passengers of average weight 160 lb., attains a speed of 60 mi./hr. in 10 min., starting from rest. If frictional resistances amount to 10 lb./T., find the draw-bar pull between the locomotive and first coach, and also between the last two coaches.

87. Two weights of 5 and 6 lb. are supported by a light cord passing over a fixed pulley which is practically frictionless. Find the tension in the cord after the weights have begun to move.

88. The arrangement of weights in the preceding problem is called an "Atwood Machine," and is used for determining the value of g, the acceleration of gravity. In an experiment of this kind the two weights were 21 oz. and 20 oz., and it was observed that in 5 sec. the heavier weight descended 9.5 ft. Find the value of g.

NOTE. — Equate the values of the acceleration found from $s = \frac{1}{2}at^2$ and from $a = \frac{g(21-20)}{21+20}$. The result, however, will not be accurate, due to the fact that friction and the resistance of the air have been neglected. For this reason Atwood's Machine is chiefly of historic interest.

89. A 10 lb. weight hangs over the edge of a table and by means of a cord pulls a 20 lb. weight along the table. If the coefficient of friction between the 20 lb. weight and the table is 0.2, find the acceleration and the tension in the cord.

90. Two cords pass over a pulley which may be regarded as frictionless. On one side both cords are attached to a weight of 12 lb. and on the other side one cord carries a weight of 6 lb., and the other 8 lb. Find the tension in each cord during motion.

91. A fixed pulley carries a light string to one end of which a weight W_1 is attached. To the other end is attached another light pulley, around which passes a string carrying weights W_2 and W_3 (Fig. 71). Determine the motion and the tensions in the strings.

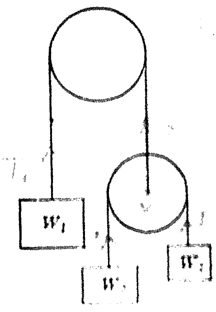


FIG. 71

92. What pressure will a man weighing 160 lb. exert on the floor of an elevator descending with an acceleration of 4 ft./sec.^{2}?

41. Work. — If a point is displaced in a straight line under the action of a force which is constant in magnitude and direction, the product of the length of the displacement and the component of the force in the direction of the displacement is called the **work** done by the force in producing the displacement. Thus in Fig. 72 let F represent a force and AB its displacement. Resolving F into components $F \cos \theta$ and $F \sin \theta$, the latter component, being perpendicular to

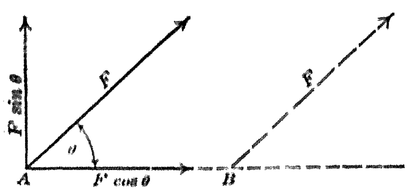


FIG. 72

the displacement, does no work. The total work done by F in the displacement AB is therefore $F \times AB \cos\theta$.

If the displacement is not rectilinear and the force is not constant, resolve the force F at any instant into rectangular components X, Y, Z, and also the element of the path ds into rectangular components dx, dy, dz. Then the element of work dW is

$$dW = Xdx + Ydy + Zdz,$$

and the total work done in a displacement along any portion of the path AB is the line integral

$$W = \int_B^A \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} \right) ds,$$

integrated along the path between the limits A and B.

Since work is defined as the product of force and distance, the unit of work is measured by unit force acting through unit distance. In the British system the unit of work is the **foot pound**. In the French system it is called the **erg**, which is defined as the work done by a force of one dyne acting through a distance of one centimeter. Larger units of work are the **megalerg**, equivalent to a million ergs, the **joule**, equivalent to 10^7 ergs, and the **kilogram-meter**, equivalent to 10^5 ergs.

In many cases of work met with in practice the displacement is rectilinear. In this case the work done by a constant force is represented graphically by the area of a rectangle, one side of which represents the displacement and the other side the component of the force in this direction (Fig. 73). If the force is variable, the only difference in the representation is that the diagram has a variable ordinate, giving a curved instead of a straight boundary, as shown in the figure.

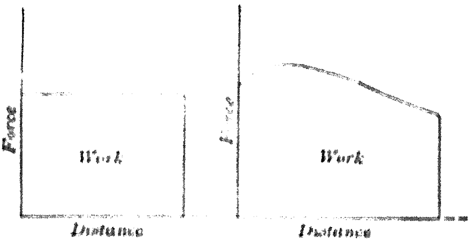


FIG. 73

The indicator diagram introduced by Watt for use on the steam engine affords one of the most simple graphical illustrations of the work done by a variable force. A sectional view of a steam

cylinder, showing its relation to the indicator diagram, is shown in Fig. 74. Laying off the steam pressures as ordinates from the line OK as axis and the corresponding displacements from the line OH, the points so determined form a closed curve $ABCDEF$. At the point E steam is admitted, the pressure remaining practically constant, as shown by the horizontal pressure line ED, until the valve closes, at D. The steam then expands adiabatically during the remainder of the stroke, and the pressure drops correspondingly, as shown by the curve DC. At C the exhaust valve opens and the pressure drops to B, following the curve BA during the return stroke. At A the exhaust valve closes and the back pressure rises to F, at which point steam is again admitted and the cycle repeated. The total work done during a complete cycle is therefore represented graphically by area $OEDCK$ − area $OEFABK$, which evidently represents the area inclosed by the curve. In engine tests indicator cards like the above are drawn automatically, and the work done is determined from them by measuring their areas by means of a planimeter. The quotient of this area of the length of the stroke is called the **mean effective pressure** (m. e. p.), and represents the average force exerted on the piston during the stroke.

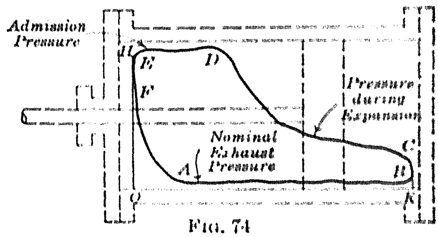


Fig. 74

The relation between force and displacement involved in the idea of work has many important practical applications, as illustrated in the following problems.

PROBLEMS

93. Find what force acting parallel to the plane is required to support a weight W on an inclined plane, without friction (Fig. 75).

Solution. Work done in moving from A to B is $F \cdot AB$. Work stored at B is $W \cdot BC$. Hence $F \cdot AB = W \cdot BC$, or since $BC = AB \sin \alpha$, $F = W \sin \alpha$.

94. A force F acting at the end of a lever of length l raises a weight W by means of a screw jack (Fig. 76). Neglecting friction, find the relation between F and W.

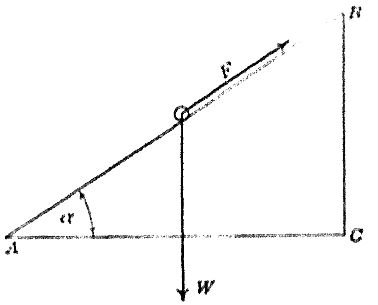


FIG. 75

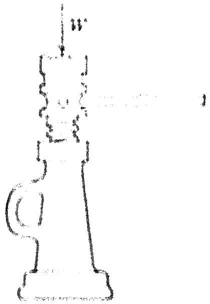


FIG. 76

SOLUTION. Work done in one revolution is $F \cdot 2\pi l$. Let p denote the pitch of the screw, where for a single-threaded screw the pitch = distance between two successive threads. Then work done on W in one revolution is Wp. Hence $F \cdot 2\pi l = Wp$, or $F = W\frac{p}{2\pi l}$.

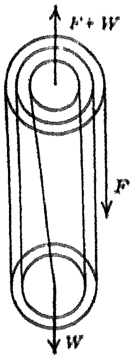


FIG. 77

95. Find the relation between the force exerted and the weight lifted in the tackle shown in Fig. 77, neglecting friction.

96. The Mallet compound locomotives used on the Erie Road weigh 410,000 lb., the entire weight being on the drivers, which are 16 in number. The maximum tractive effort (= draw-bar pull) of these locomotives is 94,800 lb., and they haul trains weighing 2600 T. up a ruling grade of 1.3 per cent. Find the frictional resistance of the train in lb. / ton.

97. In Prob. 96 find the time required to accelerate the train to 10 mi. / hr. on the level, and also the distance required to get up this speed.

98. Find the work done in hauling a train of 100 T. 2 mi. up a 1.3 per cent grade, the frictional resistance being 8 lb. / ton.

42. Principle of Work and Energy. — From Newton's laws, the fundamental dynamical relation is

$$F = ma = m\frac{dv}{dt}.$$

The element of work done by the force F in producing a displacement ds is then

$$Fds = m\frac{dv}{dt}ds = m\frac{ds}{dt}dv,$$

or, since $\frac{ds}{dt} = v$, this becomes

$$Fds = mvdv.$$

The entire work done by F on m in describing any portion of the path AB is therefore given by the integral

$$\int_B^A Fds = \int_{v_0}^{v} mvdv = \tfrac{1}{2}\, mv^2 - \tfrac{1}{2}\, mv_0^2,$$

where v denotes the speed of m at the point A, and v_0 its speed at the point B. The left member of this equation is the work done by the force F in producing the given displacement, whereas the quantity $\frac{1}{2}\, mv^2$ is called the **kinetic energy**, the first term on the right representing the kinetic energy of m at the speed v, and the second term its kinetic energy at the speed v_0.* Consequently the work done on a body by any force, either variable or constant, is equal to its change in kinetic energy. This is known as the **Principle of Work and Energy.**

This principle is of such importance that the following general demonstration is also given. Start as before from the fundamental dynamical relation

$$F = ma = m\frac{d^2s}{dt^2},$$

and resolve the force F_r acting on any particle of mass m_r into rectangular components X_r, Y_r, Z_r, where

$$X_r = m_r\frac{d^2x_r}{dt^2}; \quad Y_r = m_r\frac{d^2y_r}{dt^2}; \quad Z_r = m_r\frac{d^2z_r}{dt^2}.$$

From Art. 41 the work done by the impressed forces on any system of particles is obtained from

$$dW = \sum(X_r dx_r + Y_r dy_r + Z_r dz_r) = \sum\left(X_r\frac{dx_r}{dt} + Y_r\frac{dy_r}{dt} + Z_r\frac{dz_r}{dt}\right)dt.$$

Or, substituting the above expressions for the force components,

$$dW = \sum\left[m_r\frac{d^2x_r}{dt^2}\frac{dx_r}{dt} + m_r\frac{d^2y_r}{dt^2}\frac{dy_r}{dt} + m_r\frac{d^2z_r}{dt^2}\frac{dz_r}{dt}\right]dt.$$

* In considering the translation of a body of mass m, the entire mass of the body may be regarded as concentrated in a single particle located at the center of gravity of the body, and all the impressed forces applied to this particle, as shown in Art. 72.

To transform this relation, make use of the identity

$$m\frac{d^2x}{dt^2}\frac{dx}{dt}=\frac{m}{2}\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2\right].$$

Then the above value of dW becomes

$$dW=\sum\frac{m_r}{2}\frac{d}{dt}\left[\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right]dt.$$

Consequently the total work done during the displacement of the system is

$$\int dW=\int\sum\frac{m_r}{2}\frac{d}{dt}\left[\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right]dt.$$

Since integration is the inverse of differentiation, we have

$$\int\frac{m_r}{2}\frac{d}{dt}\left[\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right]dt=\frac{m_r}{2}\left[\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right]$$

$$=\frac{m_r v_r^2}{2}.$$

This last quantity, however, is the kinetic energy of the system. Denoting it in the initial and final positions by E_0 and E, respectively, we have finally

$$\int dW=E-E_0;$$

that is to say, the work done by the impressed forces during the displacement is equal to the change in kinetic energy of the system.

To give a simple illustration of the principle of work and energy, consider the forces acting on a traction car. Plotting the propelling force at each instant as ordinate and the corresponding distance as abscissa, a work diagram is obtained, as explained in Art. 44. Thus in Fig. 78, OA represents the force required to start the car. As the car approaches full speed the force required to maintain motion gradually decreases as shown by the curve AG. At G power is turned off and the ordinate drops to zero.

Similarly, OD represents the frictional and other resistances, the total resistance gradually increasing with the speed, as shown by the curve DB. Applying the brakes when power is turned off, the frictional resistance is thereby increased, as shown by the curve BC.

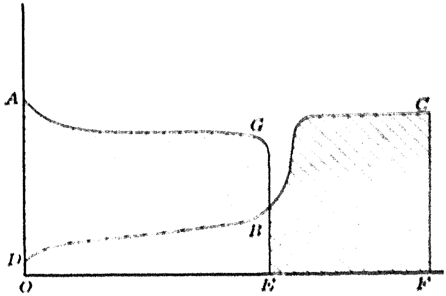


FIG. 78

The total work done upon the car by the motor is therefore represented by the area $OABE$, and the work done against friction in the interval OE by the area $ODBE$. The difference between these quantities, represented by the area ABD, is the useful work done upon the car and is therefore equal to the kinetic energy stored in the motion. In order for the car to stop, this kinetic energy must be destroyed. That is to say, the car will continue in motion until the area $BCFE$ becomes equal to the area ABD.

This illustration is typical of all practical problems in work and energy, as friction is always an important factor.

43. Transformation of Energy. — In mechanics two kinds of energy are ordinarily considered: energy of motion, or **kinetic energy**, and energy of position, or **potential energy**. The effect of doing work upon a body is to store up energy, which can again be transformed into mechanical work, as illustrated in the preceding article. Energy has therefore the properties of a material substance, since it remains constant throughout all changes, and may be stored up in one form or the other without losing its identity. The idea of energy as a substance is of comparatively recent origin, and is especially valuable in considering the relations of chemical and electrical energy to mechanical energy.

From this standpoint work appears, not as a special form of energy, but as a transformation of energy from kinetic to potential or the reverse, the amount of work done being the measure of the transformation that has taken place. Thus, by means of the proper mechanical equivalents, the transference of energy

through heat, electricity, or other means can be expressed in terms of work, which may, therefore, be regarded as the common measure of all transformations of energy.

The two forms of potential energy commonly met in mechanics are the potential energy of gravity, or energy of position, such as that of a suspended weight, and the potential energy of deformation, as in the case of a compressed spring or bent beam.

To illustrate the former, suppose a weight of W lb. is lifted through a height of h ft. Then the work done upon it, or increase in potential energy, is Wh ft.-lb. If, now, it falls freely through x feet, the velocity acquired is $v = \sqrt{2gx}$, and its remaining potential energy is $W(h-x)$. Hence its kinetic energy is $\frac{1}{2}mv^2 = \frac{W}{2g}(2gx) = Wx$, and the sum of the kinetic and potential energy is $W(h-x) + Wx = Wh$, so that no energy has been gained or lost.

A noteworthy property of energy is that it consists of the product of two factors, one being a quantity factor and the other an intensity factor. Thus in the case of heat, the quantity factor is the specific heat and the intensity factor is the change in temperature; or instead of the specific heat, the entropy may be considered as the quantity factor. In potential due to gravity, the weight is the intensity factor, and the relative height is the quantity factor. In electricity the charge, or electric capacity, is the quantity factor, and the electromotive force is the intensity factor. In kinetic energy $\frac{1}{2}mv^2$, the momentum mv is the quantity factor and the speed v is the intensity factor, etc.

The unit of energy is the same as the unit of work, the units in common use being the **foot-pound** in the British system, and the **joule** ($= 10^7$ ergs) in the French system.

PROBLEMS

99. Show that to give a velocity of 20 mi./hr. to a train requires the same energy as to lift it vertically through a height of 13.4 ft.

100. A train of 100 T. is running at 60 mi./hr. What constant force is required to bring it to rest in 2 min.?

101. A car shunted up a 1 per cent grade with a velocity of 30 mi./hr. just reaches the top of the incline. Find the time occupied, assuming frictional resistances to be 6 lb./ton.

102. An engine takes water while running at 70 mi./hr. from a trough between the rails by means of an L-shaped pipe, projecting forwards, which is let down into the trough. If there was no frictional resistance, find the height to which the water could be raised in this way.

103. A railroad station is placed upon an elevation, the grade of the track on each side being 0.5 per cent for a distance of half a mile. Find the saving effected in starting and stopping a train at the station by such an arrangement in ft.-lb./ton, assuming that the speed of approach is sufficient to utilize the entire amount.

104. A weighted screw press consists of an ordinary screw press with the addition of heavy weights on each end of the lever. When these are set in rapid rotation, they acquire kinetic energy which is utilized by the punch or die. If each of the two weights weighs 50 lb. and has a linear speed of 10 ft./sec. at the instant the punch begins to operate, find how thick a plate of metal can be punched if the diameter of the hole is $\frac{3}{4}$ in. and the average resistance is 40,000 lb./in.^{2} of the area sheared or punched.

44. Power. — Power, or activity, is defined as the rate of doing work, or the amount of work done per unit of time. Power is, therefore, the rate at which change of energy is taking place. The unit of power is given by the unit of work per unit of time. In the British system it is the foot-pound per second, abbreviated into ft.-lb./sec., and in the French system it is the erg per second.

In practical work, a larger unit than the ft.-lb./sec. is desirable. The enlarged unit in common use is the **horsepower,** abbreviated into h. p., which is supposed to represent the rate at which an ordinary draft horse works. From experiments made by James Watt in London it was found that an ordinary horse could walk at the rate of $2\frac{1}{2}$ mi./hr. and at the same time raise a weight of 100 lb. from the bottom of a shaft by a rope passing over a pulley. This is equivalent to $2.5 \cdot \frac{5280}{60} \cdot 100 = 22{,}000$ ft.-lb./min. By adding 50 per cent to this for work lost in friction, etc., the average value of a horsepower was determined as

$$\textbf{One h. p.} = \textbf{33,000 ft.-lb./min.} = \textbf{550 ft.-lb./sec.}$$

More recently General Morin has estimated that the average power of a horse is only 26,150 ft.-lb./min.

From the definition of power as the rate of doing work, the

power exerted by a force F which performs the element of work Fds in the time dt is

$$\text{Power} = F\frac{ds}{dt} = Fv.$$

In the solution of practical problems it is especially convenient to remember this relation in the form

$$\text{h. p.} = \frac{Fv}{550},$$

where F is expressed in pounds and v in ft./sec.

In electrical engineering the unit of power commonly used is the **watt**, equivalent to 10^7 ergs/sec., or the **kilowatt**, equivalent to 1000 watts. The relation between the two units, watt and horsepower, is

$$1 \text{ h. p.} = 746 \text{ watts},$$

or, roughly, 4 h. p. = 3 kw.

The horsepower is also used in France (*Force de cheval*) and in Germany (*Pferdestärke*), but is not precisely equivalent to the British horsepower. The metric horsepower used on the continent = 75 mkg./sec., whereas the British horsepower of 550 ft.-lb./sec. is equivalent to 76.04 mkg./sec.

PROBLEMS

105. In an hydraulic turbine installation, the pipe lines deliver 335 cu. ft./sec. to each wheel, with an available head of 365 ft. Calculate the theoretic horsepower available.

106. In Prob. 105 the running portion of the wheel is 62 in. in diameter and makes 376 r. p. m. How does the linear speed of a point on the rim compare with that possible from the given head of water, neglecting pipe-line friction?

107. Show that the hydraulic power developed is given by the formula

$$\text{h. p.} = 0.1134\, QHE,$$

where
Q = discharge in cu. ft./sec.,
H = head in feet,
E = efficiency of conversion,

and from this determine what quantity of water per second under an effective head of 25 ft. is necessary to deliver 400 h. p. at the switchboard of a water-wheel plant which has a total efficiency of conversion of 50 per cent.

108. What power will be available at the shaft of a water-wheel operated by a flow of 120 cu. ft./sec. under an effective head of 50 ft., assuming that the pipe-line and water-wheel losses are 25 per cent?

109. In a commercial test of a Pelton water-wheel (see Fig. 38, Art. 39), the diameter of the jet = 1.89 in., actual head = 386.5 ft., head lost in friction = 1.8 ft., reducing the effective head to 384.7 ft. The actual amount of water discharged by the jet was found by measurement to be 2.819 cu. ft./sec. The h. p. developed by the wheel in this test was found to be 107.4. Calculate the efficiency of the wheel.

110. What electric h. p. can be delivered at the switchboard of a water-power plant which has a water supply of 200 cu. ft./sec. under an effective head of 75 ft. if the water-wheel and pipe-line losses are 25 per cent, and the generator and station losses are 20 per cent?

111. Find the useful work done per second by a fire engine which discharges water at the rate of 500 gal. per minute against a pressure of 100 lb./in.^{2}.

Note. — 1 gallon of water weighs $8\frac{1}{3}$ lb., and 1 lb./in.^{2} pressure = 2.304 ft. of head.

112. A 14×20 engine running at 240 r. p. m. has a 60 lb. indicator spring, with a reducing motion which gives a $3\frac{1}{2}$ in. card. If the area of the card is 2.44 in.^{2}, find the i. h. p.

Note. — A 14×20 engine is one with cylinder 14 in. in diameter and 20 in. stroke. The length of the indicator card (Fig. 74) is in this case $3\frac{1}{2}$ in. Dividing the area of the card in square inches by its length gives the mean effective pressure in inches. Since the spring in this case is such that it requires 60 lb. force to produce 1 in. of deflection, the m. e. p. in pounds is obtained by multiplying its value in inches by 60. The indicated horsepower may then be found from the formula (Art. 45)

$$\text{i. h. p.} = \frac{PLAN}{33000}.$$

113. A 5 h. p. steam hoist raises a load of 12 T. to a height of 85 ft. in 19 min. Find what proportion of the work expended is wasted in friction.

114. A milling machine has a driving pulley 16 in. in diameter with a 5 in. belt and runs at 300 r. p. m. The motor is 10 h. p. Find the tension in the driving side of the belt if the following side runs slack.

115. The milling machine shown in Fig. 79 is taking a roughing cut across 20 point carbon steel bars with a cutter 12 in. in diameter, running at 17 r. p. m. The gross h. p. used is 12.1, which when corrected for motor efficiency gives 10 h. p. net. The cut is 6 in. wide and $\frac{1}{4}$ in. deep, and the feed is 6.7 in. per minute. Find the number of cubic inches of metal removed per net h. p. minute.

116. In a hill-climbing contest an automobile weighing 1 T. acquired a speed from rest of 45 mi./hr. up a 17 per cent grade in a distance of 250 ft. Neglecting friction and air resistance, find the h. p. of the engine.

117. A belt is designed to stand a difference in tension of the two sides of 100 lb. only. Find the least speed at which it can be driven to transmit 20 h. p.

45. Measurement of Power. — The rate at which a prime mover is doing work, whether it be a boiler, engine, dynamo, turbine, or machine, is called its **indicated power.**

Boiler power is determined by the capacity of the boiler for producing steam, and in reality refers to the horsepower of the engine which the boiler is capable of running. The standard rating in this country is

One boiler horsepower = 34.5 lb. of water evaporated per hour from and at 212° F.

By this is meant simply the conversion into steam of 34.5 lb. of water at the boiling point, without considering the amount of heat required to raise it to this temperature.* Since boilers are usually sold without being tested as to their steam capacity, they are frequently rated according to the number of square feet of heating surface. This varies largely with the style and make of boiler, average values being from 10 to 12 sq. ft. per horsepower for water tube boilers.†

The indicated horsepower of an engine is obtained from the indicator diagram. The area of this diagram divided by its length gives the mean effective pressure P in lb./in.², as explained in Art. 41. Then if A denotes the area of the piston in square inches, the average pressure on it is PA, and hence if L denotes the length of the stroke, the work done in one stroke is PLA. Consequently if the number of strokes per minute is denoted by N, the horsepower of the engine is

$$\text{h. p.} = \frac{PLAN}{33000}.$$

* This rating dates from the Centennial Exposition at Philadelphia in 1876, where, in order to secure uniformity, the judges decided that one boiler horsepower should mean 30 lb. of water evaporated per hour from an initial temperature of 100° F. under a gauge pressure of 70 lb./in.² This is equivalent to 34½ lb. per hour from and at 212° F.

† The grate area also varies largely with the style of boiler, kind of fuel, and nature of the draft, whether natural or forced. An average value of the grate area for tubular boilers is ⅓ sq. ft. per horsepower.

For natural draft the size and height of stack depends on the number of square feet of grate area. A good practical rule is to make the area of the stack one seventh the grate area, and its height twenty-five times its inside diameter.

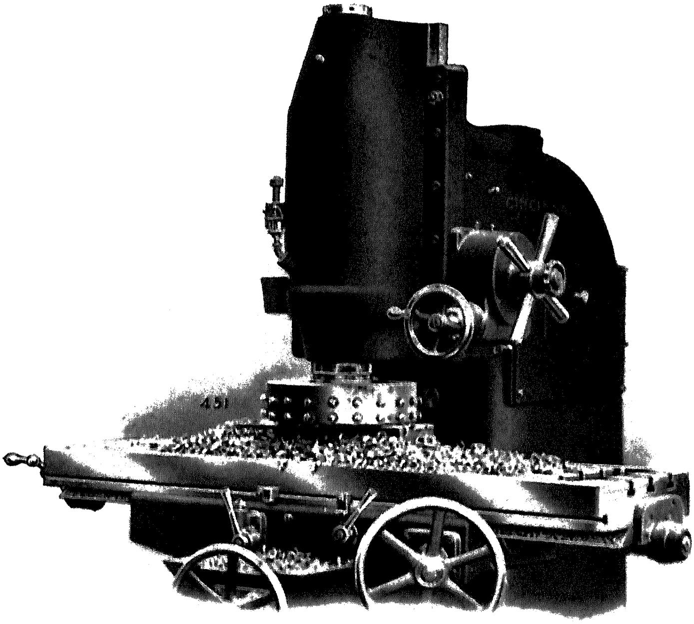


FIG. 79

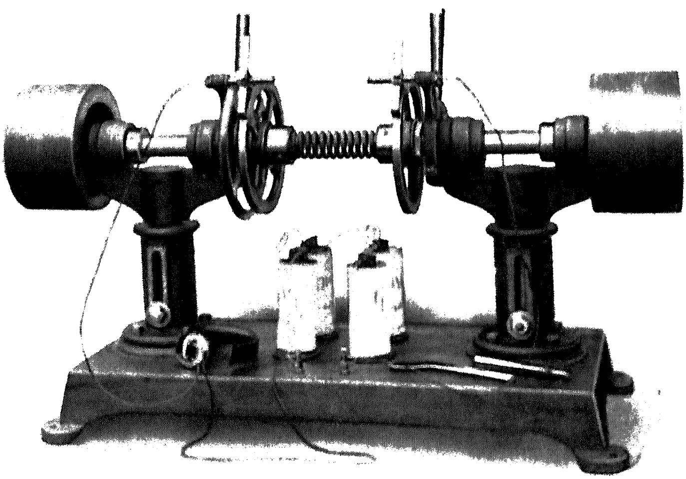


FIG. 82. — Transmission Dynamometer

In electric transmission of energy, the method to be used in determining the indicated power of an alternating current depends on whether the circuit is inductive or non-inductive. In a non-inductive circuit it is simply necessary to measure the current in amperes by means of an ammeter, and the pressure, or electromotive force, in volts by means of a voltmeter. The indicated power in watts is then given by

$$\textbf{watts} = \textbf{volts} \times \textbf{amperes}.$$

When the inductance of the circuit is considerable, the power is determined by means of an electrodynamometer or wattmeter. Of these there are two types, the indicating and the integrating. The former simply indicates the power being transmitted at any given instant, whereas the latter records the total power which has passed through the meter.

To measure the power actually delivered a number of devices are in use, one of the simplest being the Prony brake. This consists of a simple lever, one end of which is clamped to the engine shaft, and a weight W attached to the other end (Fig. 80). By tightening the screws at A and B friction is developed, which tends to cause the whole apparatus to rotate with the shaft in the direction shown by the arrow, whereas the weight W has a tendency to cause it to rotate in the opposite direction. Hence, by tightening the screws at A and B and thereby increasing the friction, and at the same time adjusting the weight W so that the lever remains in a horizontal position, the brake may be so arranged that the entire power of the engine is absorbed by friction. Such a device is called an absorption dynamometer.

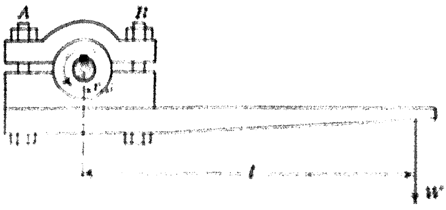


FIG. 80

Let r denote the radius of the shaft, F the frictional force acting at its circumference, W the weight, and l its distance from the center of the shaft (Fig. 80). Then the motion of the shaft relative to the brake is the same as though the shaft was at rest and the brake rotating around it. In the latter case, the work done by

F and W in one revolution would be $2\pi rF$ and $2\pi lW$, respectively. Hence, $2\pi rF = 2\pi lW$ or $rF = lW$. If, then, the shaft is revolving uniformly n times per minute, the work done by the engine in one minute is equal to the work done by the friction in n revolutions, or $2\pi rnF$. Since $F = \frac{Wl}{r}$, this expression becomes

$$\text{work per minute} = 2\pi nWl.$$

Hence, if W is expressed in pounds, l in feet, and n in r. p. m., then since 1 h. p. = 33,000 ft.-lb. / min., the horsepower of the engine is given by the formula

$$h.p. = \frac{2\pi nWl}{33000} = 0.00019\, nWl.$$

A somewhat different form of absorption dynamometer, consisting of a flexible metal strap with wood lagging, applied to a flywheel or pulley, is shown in Fig. 81. Here the scales take the place of the weight W, and hence the motion is in the opposite direction to that indicated in Fig. 80. Otherwise the relations are the same as above.

Recently a new form of dynamometer has been introduced, consisting of a spiral spring clamped to chucks or pulleys, as shown in Fig. 82. By belting from the driver to one pulley of the dynamometer, and from the other pulley to the machine to be driven, the power is transmitted through the spring, which is twisted through an angle proportional to the torque. Thus, if T denotes the torque in ft.-lb. and θ the angle of twist in radians, we have

$$T = k\theta,$$

where k is a constant which is obtained experimentally for each dynamometer. For n revolutions per minute the work done per minute is $2\pi nT$, and consequently the horsepower transmitted is

$$\text{h. p.} = \frac{2\pi nT}{33000} = \frac{2\pi nk\theta}{33000}.$$

Let c denote the constant part of this expression; namely,

$$c = \frac{2\pi k}{33000}.$$

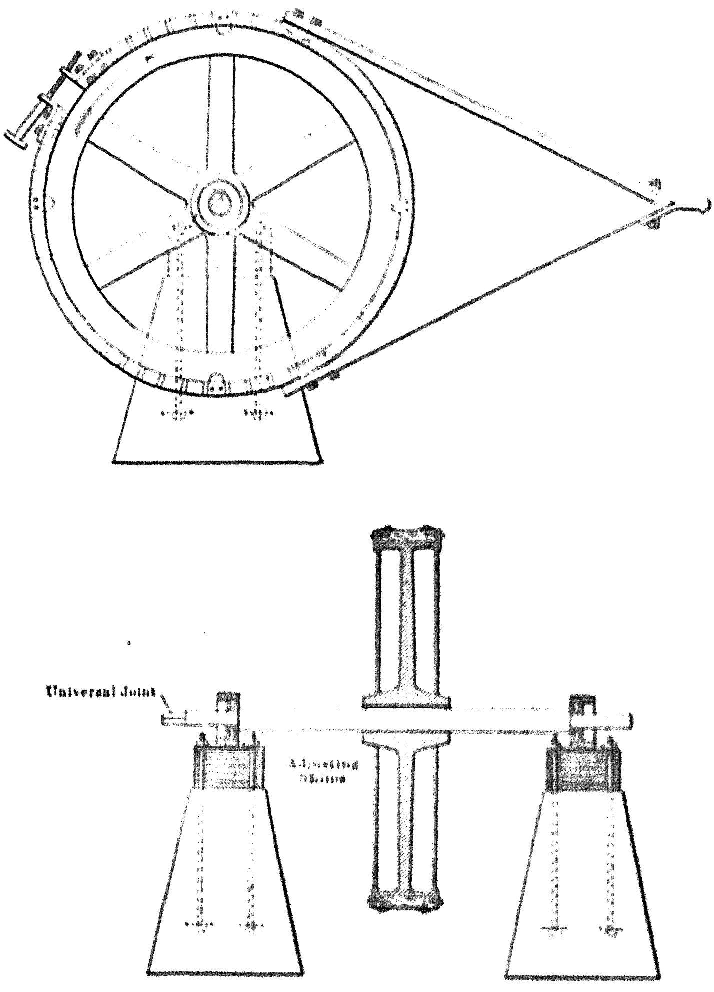


Fig. 81. — Prony Brake of Large Capacity

Then the formula for calculating the power transmitted becomes simply

$$\text{h. p.} = cn\theta,$$

where n is the number of r. p. m., determined by means of a speed indicator, and θ is read from the dynamometer by means of a battery and telephone receiver connected in series through the brushes shown in the figure.

The makers claim for this type of dynamometer that it is more sensitive and accurate than the Prony brake, has a wide range of power and speed, is easily operated, and is subject to practically no wear.

PROBLEMS

118. The following data were determined experimentally for various sizes of spiral spring dynamometers. Determine the constant c for each type.

Diam. Helix Inches	Diam. Wire Inches	Number of Turns	Turns per Inch	Angular Deflect. Degrees	Torque Foot-pounds
$1\frac{1}{2}$	0.122	16	2	25	1.11
$1\frac{1}{4}$	0.250	$15\frac{3}{4}$	2	25	23.8
$1\frac{9}{32}$	0.3125	$15\frac{3}{4}$	2	25	27.5
$1\frac{9}{32}$	0.3125	$15\frac{3}{4}$	2	25	50.5
$1\frac{1}{4}$	0.375	$15\frac{3}{4}$	2	10	51.25

119. In a boiler test lasting $7\frac{1}{2}$ hr., 71,171.8 lb. of water were evaporated with 9235 lb. of coal. Average temperature of feed water was 189° F. Average current developed by dynamo was 430 amperes at an average pressure of 243.5 volts. Find the boiler horsepower and indicated electric horsepower.

Solution. Since the feed water was supplied at 189° F., the number of pounds of water evaporated must be corrected for the difference in temperature between 189° and 212°. The correction factor is

$$c = \frac{H - (t - 32)}{l},$$

where H denotes the total heat of evaporation above 32°, in heat units, and l is the latent heat of water at atmospheric pressure = 966 heat units.

Substituting $H = 1180$, $t = 189°$, and $l = 966$, the correction factor in the present case is $c = 1.06$, which shows that if the feed water had been supplied at 212° F. instead of 189° F. we should have had heat enough to evaporate 1.06 lb. of water for every pound of water evaporated under the given conditions. Applying this correction, we have $71{,}171.8 \times 1.06 = 75{,}442$ lb. from and at 212° F. Hence $\frac{75442}{9235} = 8.17$ lb. water evaporated per pound of coal. Also $\frac{75442}{7.5} = 10{,}059$ lb. water evaporated per hour, and hence $\frac{10059}{34.5} = 291.6$, boiler horsepower. The indicated electric power

$$= \frac{430 \text{ amp.} \times 243.5 \text{ volts}}{746} = 140 \text{ h. p.}$$

120. The formula adopted by the Association of Licensed Automobile Manufactures for the commercial rating of motors is

$$\text{h.p.} = \frac{D^2 \times N}{2.5},$$

where D is the bore of the cylinder in inches, N the number of cylinders, and 2.5 an arbitrary constant, based on 1000 ft./min. piston speed. The same formula has also been adopted by the Royal Automobile Club of Great Britain and Ireland, and is used as a basis for nearly all contests held abroad.

Find from this formula the commercial rating of a four-cylinder motor of $4\frac{1}{4}$ in. bore and $4\frac{1}{2}$ in. stroke.

46. Efficiency. — In any machine for doing work, the amount of work obtained from the machine is always less than the amount put into it. This loss is due to friction and other causes, the effect of which is to dissipate energy in the form of heat, wear, etc. For example, in a locomotive only a small fraction of the heat energy in the fuel is actually converted into steam pressure, the greater part being lost by the escape of smoke and gases from the smoke stack and by radiation from the fire box. Part of the energy in the steam is also lost by radiation from the walls of the boiler and cylinders, and part of what remains is used in overcoming the frictional resistance of the moving parts, so that only a small amount of energy is left to be converted into useful work.

The ratio of the useful work obtained from an engine to the heat energy stored in the fuel it consumes is called the **duty** of the engine; that is to say,

$$\textbf{Duty of engine} = \frac{\textbf{useful work}}{\textbf{heat energy in fuel}}.$$

This fraction is of course always less than unity, the difference between numerator and denominator being the loss due to all causes, such as friction, radiation, imperfect combustion, etc.

Similarly the efficiency of any prime mover, or machine, is determined by the ratio of the useful work obtained from it to the total work done by it; that is to say,

$$\textbf{Mechanical efficiency} = \frac{\textbf{output}}{\textbf{input}}.$$

This fraction is also less than unity, but is larger than the duty, as it includes fewer losses. Since power is the quantity usually

measured in determining the performance of a machine, the efficiency is ordinarily determined from the ratio of the braked horsepower to the indicated horsepower; that is,

$$\text{Efficiency} = \frac{\text{b. h. p.}}{\text{i. h. p.}}.$$

The following problems illustrate numerical values of efficiency as actually found in practice.

PROBLEMS

121. In the builder's tests of the U. S. scout cruiser *Salem*, equipped with Curtis turbines, the data from three tests at different speeds were as follows:

(*a*) Mean speed for four hours, 25.947 knots; b. h. p., 19,200; coal used per hour, 38,502 lb.

(*b*) Mean speed for twenty-four hours, 22.536 knots; b. h. p., 9340; coal used per hour, 18,485 lb.

(*c*) Mean speed for twenty-four hours, 11.93 knots; b. h. p., 1360; coal used per hour, 4051 lb.

Find the number of knots per ton of coal at each speed, and the relative efficiency. Find also the resistance to motion at each speed, and show how this resistance varies with the speed.

122. Two tests of a boiler feed pump gave the following data:

1. Number of strokes per minute	20.00	72.20
2. Water pumped, gal./min.	38.00	139.00
3. Piston displacement, gal./min.	40.00	145.00
4. Rated displacement, gal./min.	41.90	154.20
5. i. h. p., steam end	2.25	8.80
6. i. h. p., water end	2.10	8.26
7. Developed h. p.	2.00	8.00
8. b. t. u. per d. h. p.	3965	1890

Compare the efficiency calculated from the ratio $\frac{\text{i. h. p. water cylinder}}{\text{i. h. p. steam cylinder}}$ with that calculated from the ratio $\frac{\text{i. h. p. steam cylinder}}{\text{d. h. p.}}$.

Also calculate the per cent of slip from the ratios $\frac{\text{line 3} - \text{line 2}}{\text{line 3}} \times 100$, and $\frac{\text{line 4} - \text{line 3}}{\text{line 4}} \times 100$.

123. In a duty test of a pumping engine for a flushing plant at Milwaukee in April, 1908, the data were as follows:

Steam pressure 140 lb./in.^{2}; r. p. m. 53; evaporation per pound of coal 7.18; water displacement per revolution 586.77 cu. ft.; theoretic displacement of wheel per revolution without slip 740.13 cu. ft.; i. h. p. of engine 315.42; b. h. p. from work performed 209.96. Calculate the efficiency of the wheel and the efficiency of the pumping machinery.

124. In the preceding problem the amount of coal burned per hour was 520.81 lb. Find the duty of the engine in foot-pounds of useful work per 100 lb. of coal consumed. Also, if each pound of coal contains 10,000 B. t. u., calculate the efficiency.

NOTE. — 1 h. p.-hr. $= \dfrac{33000 \times 60}{778} = 2545$ B. t. u.

125. The following table gives the performance of the cars winning the first four places in the fourth Vanderbilt Cup Race, October 24, 1908. Total distance 258.06 mi.

Car	H. P.	Lap Time in Minutes and Seconds 1st Lap	2d Lap	3d Lap	4th Lap	5th Lap	6th Lap	7th Lap	8th Lap	9th Lap	10th Lap	11th Lap	Total Time
Locomobile	120	20.54	22.13	20.17	25.50	22.23	20.30	20.30	20.36	22.22	20.55	24.12½	240.48½
Isotta	60	21.52	21.34	21.49	22.04	23.28	22.00	22.05	21.50	21.40	22.39	21.50½	242.36½
Mercedes	120	23.82	25.44	22.54	22.24	22.16	22.25	27.27	22.07	21.25	27.40	[illegible]	256.35
Locomobile	120	23.10	22.18	22.13	20.08	21.38	28.17	21.36	25.59	30.09	21.44	22.51	253.10

Find the greatest speed in miles per hour attained by any car. Assuming that the Isotta motor was working at its full rated power, find the total resistance to motion. Assuming that the average resistance to motion while getting up speed was two thirds of this amount, and that the winning Locomobile weighed one ton and attained its maximum speed in 10 seconds from rest (first lap 20.54 − minimum time 20.17 = 37 sec.), compute the efficiency of its motor at starting and at full speed.

126. In the gas engine plant of the Somerville Power Station of the Boston Elevated Railway Company the amount of coal consumed was 2.031 lb./kw.-hr. In a steam plant the amount used was in one case 3.477 lb./kw.-hr., and in another case 4.414 lb./kw.-hr. What is the relative efficiency of the gas and steam plants?

127. What h. p. motor should be installed for a passenger elevator service which will have an unbalanced load of 3000 lb. and speed of travel of 250 ft./min. if driving efficiency is 90 per cent, motor efficiency 85 per cent, and worm gear efficiency 60 per cent?

128. How many pounds of unbalanced weight will a freight elevator driven by a 60 h. p. motor lift at 150 ft./min. if worm gear efficiency is 65 per cent and driving efficiency 80 per cent?

47. Principle of Impulse and Momentum.—The effect of a force may be measured either by the product of the force by the distance through which it acts, or by the product of the force by the length of time it acts. The first product is called the work, as explained in Art. 41, whereas the second is called the **impulse.** Thus if a constant force F acts for t seconds, the impulse of the force is Ft.

If the constant force F has during the interval of time t acted without resistance upon a mass m, an acceleration a has been produced in accordance with the dynamical relation $\frac{F}{m} = a$, and the speed of the body has thereby been altered from v_0 to v in accordance with the kinematical relation $a = \frac{v - v_0}{t}$ (Eq. (1), Art. 12).

Hence
$$\frac{F}{m} = \frac{v - v_0}{t},$$

or
$$Ft = mv - mv_0.$$

The product of mass and velocity is called the **momentum** of the body, mv denoting its momentum at the velocity v and mv_0 at the velocity v_0. This relation therefore expresses the fact that the change in the momentum of a body is equal to the impulse producing it.

This relation also applies to the action of a variable force. To prove this consider the action of a force, either variable or constant, upon a rigid body of mass m. Then in accordance with the fundamental relation obtained from Newton's laws,

$$F = ma = m\frac{dv}{dt},$$

or
$$Fdt = m\,dv.$$

Integrating this expression between any two time limits t and t_0, at which the velocities are v and v_0 respectively, we have

$$\int_{t_0}^{t} Fdt = \int_{v_0}^{v} mdv = mv - mv_0.$$

Consequently the impulse communicated to a body is equal to its change in momentum. This is called the **principle of impulse and momentum.**

There are many instances in dynamical problems in which the action of a force begins and ends within such a brief interval of time that its action is practically instantaneous; as, for example, in the action of a jet of steam upon the blades of a high speed turbine, revolving, say, at 15,000 r.p.m. The force in this case is called an impulsive force. The change in velocity produced is, in general, finite. Therefore, since the interval of time is infinitesimal, the acceleration, or time rate of change of the velocity, must be infinite. Hence by Newton's second law, the force must also be regarded as infinite while it lasts. An impulsive force must therefore be considered as an infinite force exerted for an infinitesimal length of time. The impulse in this case cannot be measured directly, as in the preceding demonstration, in which the interval of time is assumed to be finite, but is expressed by the change in momentum produced by it.

There is one peculiarity which is characteristic of the action of impulsive forces. A rigid body may be defined as one which does not appreciably change its shape under the action of finite forces. No body, however, can be considered as rigid when subjected to the action of an infinite force. Hence when impulsive forces are brought into action, relative motion is set up between the small particles or molecules of which the body is composed, the work done by these molecular forces appearing in the form of heat, deformation, sound, etc. Since this work is subtracted from the mechanical energy of the system, it is apparent that under the action of impulsive forces, the sum of the kinetic and potential energy does not remain constant.

Let the impulse of a force be denoted by I; that is, let

$$I = \int F dt = m(v - v_0).$$

Then the work done by the impulse in changing the velocity of a body of mass m from v_0 to v is $\frac{1}{2} mv^2 - \frac{1}{2} mv_0^2 = \frac{1}{2} m(v - v_0)(v + v_0)$. Since $m(v - v_0) = I$, the expression for the work done by the impulse may be written $I\left(\frac{v + v_0}{2}\right)$. Consequently the work done by an impulse is equal to the impulse multiplied by the mean of the initial and final velocities of the mass acted upon.

If the force is constant, the acceleration is uniform and the mean of the initial and final velocities becomes the average velocity during the interval.

PROBLEMS

129. A fireman holds a hose from which a jet of water 1 in. in diameter issues at a velocity of 80 ft./sec. What force will the fireman have to exert to support the jet?

130. A bullet weighing 2 oz. leaves the barrel of a gun 3 ft. long with a velocity of 1200 ft./sec. Find the average force exerted on the bullet during the discharge.

131. A jet of water 1 in. in diameter impinges directly on a flat plate with a velocity of 40 ft./sec. and flows off at right angles. Find the pressure exerted on the plate.

132. A forging hammer weighing 1 T. falls through 3 ft. If the force of the blow is expended in 0.02 sec., find its average value.

133. A bullet weighing 2 oz. strikes a block of wood with a velocity of 1200 ft./sec. and enters it to the depth of 10 in. Find the average resistance of the wood to penetration.

134. A machine gun fires 300 bullets per minute, each bullet weighing 1 oz. If the bullets have a velocity of 1500 ft./sec., find the average reaction of the gun against its support.

135. Water flowing in a pipe 100 ft. long with a velocity of 30 ft./sec. is shut off in $\frac{1}{10}$ of a second by means of a valve. Find the increase in pressure near the valve.

136. A series of equal elastic balls are hung by threads in a row in contact with one another. If the ball at one end is pulled off and let fall against the others, show that all will remain at rest except the ball at the other end of the row, which will fly off.

137. A 10 lb. weight falls from a height of 15 ft. upon a coiled spring and is brought to rest in $\frac{1}{20}$ of a second. Find the average and the maximum compressive force exerted on the spring.

138. A 40 T. gun discharges a 500 lb. shot with a velocity of 1600 ft./sec. If the recoil is resisted by a constant pressure of 10 T., how far will the gun recoil?

48. Conservation of Linear Momentum. — An important application of the principle just derived occurs in the mutual action of two bodies during collision or impact. In this case from Newton's third law the action of A on B is equal to the action of B on A. Since, moreover, the action and reaction are exerted for the same

interval of time, the total impulse communicated by A to B must be exactly equal to that communicated by B to A. Hence the momentum gained by one is equal to that lost by the other, and consequently the total change in momentum is zero. This is a special case of what is known as the principle of the conservation of linear momentum.

To illustrate, suppose that two bodies of masses m_1 and m_2, moving in opposite directions with velocities v_1 and v_2, collide. Their velocities are changed by the impact and become, say, v_1' and v_2'. The change in momentum of the first body is, then, $m_1v_1 - m_1v_1'$, and since the second body is moving in the opposite direction, its change in momentum is $m_2v_2' - m_2v_2$. But from Art. 47 the change of momentum in each case is equal to the impulse $\int Fdt$. Hence

$$m_1v_1 - m_1v_1' = \int Fdt = m_2v_2' - m_2v_2,$$

or, transposing,

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'.$$

Stated in words, this relation is expressed by saying that *the sum of the momenta before impact is equal to the sum of the momenta after impact.*

To obtain a more general proof of the principle of the conservation of linear momentum, consider a system of particles, and let x_r, y_r, z_r denote the three rectangular coördinates of any particle of mass m_r at a given instant, and X_r, Y_r, Z_r the components of the force acting on it. Then by addition, the equations of motion for the entire system become

$$\sum X_r = \sum m_r\frac{d^2x_r}{dt^2};\ \sum Y_r = \sum m_r\frac{d^2y_r}{dt^2};\ \sum Z_r = \sum m_r\frac{d^2z_r}{dt^2}.$$

Denoting the components of the velocity, parallel to the coördinate axes, of any particle m_r by $v_r^{(x)} = \frac{dx_r}{dt}$, $v_r^{(y)} = \frac{dy_r}{dt}$, $v_r^{(z)} = \frac{dz_r}{dt}$, we have

$$m_r\frac{d^2x_r}{dt^2} = m_r\frac{d}{dt}\left(\frac{dx_r}{dt}\right) = m_r\frac{dv_r^{(x)}}{dt} = \frac{d}{dt}(m_rv_r^{(x)}).$$

Similarly $m_r\frac{d^2y_r}{dt^2} = \frac{d}{dt}\left(m_rv_r^{(y)}\right)$; $m_r\frac{d^2z_r}{dt^2} = \frac{d}{dt}(m_rv_r^{(z)})$.

Hence the equations of motion of the system may be written

$$\sum X_r = \sum \frac{d}{dt}(m_r v_r^{(x)});$$

$$\sum Y_r = \sum \frac{d}{dt}(m_r v_r^{(y)});$$

$$\sum Z_r = \sum \frac{d}{dt}(m_r v_r^{(z)});$$

or, interchanging the signs of summation and differentiation,

$$\sum X_r = \frac{d}{dt}\sum m_r v_r^{(x)}; \quad \sum Y_r = \frac{d}{dt}\sum m_r v_r^{(y)}; \quad \sum Z_r = \frac{d}{dt}\sum m_r v_r^{(z)}.$$

These equations state that the component of the force acting on the system in any given direction is equal to the time rate of change of the corresponding component of the linear momentum of the system.

When there are no external forces acting on the system, $\sum X = 0$, $\sum Y = 0$, $\sum Z = 0$. In this case therefore we also have

$$\frac{d}{dt}\sum m_r v_r^{(x)} = 0; \quad \frac{d}{dt}\sum m_r v_r^{(y)} = 0; \quad \frac{d}{dt}\sum m_r v_r^{(z)} = 0,$$

and integrating, each component of the total linear momentum is constant. Consequently the resultant linear momentum of the system, regarded as a vector, is constant, which is the required principle of the conservation of linear momentum. Stated in words, it says that when any system of particles moves without being acted on by external forces, the total linear momentum of the system remains constant in both magnitude and direction.

PROBLEMS

139. A cannon weighing 30 T. projects a shot weighing 1000 lb. with a velocity of 1500 ft./sec. With what velocity will the cannon recoil?

140. A bullet weighing 1 oz. and moving at 1000 ft./sec. strikes a block of wood weighing 10 lb. Find the velocity with which the bullet and the block will move off, and having found this velocity, compute the loss of energy.

141. A loaded car weighing 40 T. runs into an empty car at rest and weighing 10 T., and the two move off together with a velocity of 4 ft./sec. Find the velocity of the loaded car before impact.

49. Impact. — When two bodies collide, it is found by experiment that the difference of their velocities after impact bears a constant negative ratio to the difference of their velocities before impact. This constant ratio is called the **coefficient of restitution,** and depends on the nature of the bodies, varying between the limits zero, for inelastic substances like putty, and nearly unity for substances almost perfectly elastic, such as ivory and glass. Let v_1, v_2 denote the velocities of the bodies before impact, v_1', v_2' their velocities after impact, and e the coefficient of restitution. Then this relation is expressed by the equation

$$v_1' - v_2' = -e(v_1 - v_2).$$

From the principle of the conservation of linear momentum we have another relation; namely,

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'.$$

Solving these two equations simultaneously, the velocities after impact are found to be

$$v_1' = \frac{m_1v_1 + m_2v_2 - m_2e(v_1 - v_2)}{m_1 + m_2},$$

$$v_2' = \frac{m_1v_1 + m_2v_2 + m_1e(v_1 - v_2)}{m_1 + m_2}.$$

The deformation produced by impact causes dissipation of mechanical energy, although there is no loss of momentum. To determine the amount of mechanical energy lost, let E_1 denote the total kinetic energy before impact, and E_2 the kinetic energy after impact. Then

$$E_1 = \tfrac{1}{2} m_1v_1^2 + \tfrac{1}{2} m_2v_2^2,$$

$$E_2 = \tfrac{1}{2} m_1v_1'^2 + \tfrac{1}{2} m_2v_2'^2,$$

and consequently the loss of energy is

$$E_1 - E_2 = (1 - e^2)\frac{m_1m_2}{m_1 + m_2}\frac{(v_1 - v_2)^2}{2}.$$

If the bodies are perfectly elastic, $e = 1$ and the right member becomes zero, whence $E_1 = E_2$.

The energy apparently lost during impact is in reality transformed into sound, heat, etc., or used in producing permanent deformation of the bodies.

PROBLEMS

142. The ram of a pile driver weighs W lb. and falls through a height of h ft. upon a pile weighing P lb. If the blow causes the pile to sink d ft. into the ground, find the resistance offered by the ground, or the "bearing power" of the pile.

Solution. Considering the ram and pile inelastic, $e = 0$, and the velocity of the pile before impact is $v_2 = 0$. The velocity of the ram before impact is $v_1 = \sqrt{2\,gh}$, and after impact both ram and pile have the same velocity $v_1' = v_2'$. Hence

$$v_1' = v_2' = \frac{m_1 v_1}{m_1 + m_2} = \frac{W\sqrt{2\,gh}}{W + P}.$$

The kinetic energy of ram and pile causing penetration is, therefore,

$$\frac{1}{2}\,\frac{W + P}{g}\,v_1'^2 = \frac{W^2 h}{W + P}.$$

Also the additional work done by ram and pile during penetration is $(W + P)\,d$. Hence if R denotes the resistance of the ground,

$$Rd = \frac{W^2 h}{W + P} + (W + P)\,d.$$

At the last blow the value of d is usually small, and hence the second term may be neglected in comparison with the first in which case,

$$R = \frac{W^2 h}{(W + P)\,d}.$$

Still more approximately, by neglecting the weight of the pile in comparison with that of the ram,

$$R = \frac{Wh}{d}.$$

The empirical formulas in common use are, for drop hammer pile drivers, $R = \dfrac{2\,Ph}{d + 1}$, and for steam hammers, $R = \dfrac{2\,Ph}{d + 0.1}$, where Ph denotes the kinetic energy of the hammer, h and d in these formulas, both being expressed in inches. These are commonly known as Wellington's formulas, or the Engineering News formulas.

143. Find the safe load for a pile weighing 300 lb. if it sinks half an inch at the last blow of a pile driver weighing 500 lb., falling through 6 ft.

144. A ball falls from a height of 20 ft. above a level floor and rebounds to a height of 17 ft. Find the coefficient of restitution.

145. A ball falls from a height h above a level floor. Find the total distance traversed before it comes to rest and the total time taken.

HINT. — The total distance is the sum of the series $h + 2he^2 + 2he^4 + \dots$. Derive this series. Find its sum and then find the time.

146. A ball moving with velocity u strikes a plane at an angle θ. Find its velocity v after impact, and the direction of its motion.

HINT. — Equate the normal and tangential components of the velocities before and after impact, and solve the resulting equations for v and the angle of reflection ϕ.

147. What do the above values of v and ϕ become for the special cases that (1) the impact is direct (*i.e.* $\theta = 90°$) and (2) that the elasticity is perfect (*i.e.* $e = 1$)?

148. Show that to hit a ball A with a ball B after reflection from the edge of a billiard table, the ball B should be aimed at a point as far behind the edge of the table as the ball A is in front of it.

50. Fundamental Equation for Rotation. — The simplest case of motion, next to that of translation, is a rotation about a fixed axis.

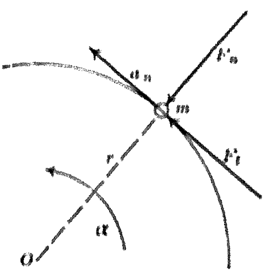


FIG. 83

Consider a rigid body, or system of particles rigidly connected, constrained to rotate about a fixed axis. Let m denote the mass of any particle of the body, or system of particles, and r its distance from the fixed axis of rotation, and let the force acting on m be resolved into components F_t and F_n, tangent and normal to the path respectively (Fig. 83). The tangential component F_t is then the effective force, producing motion along the path. Hence if a_t denotes the tangential or linear acceleration along the path, we have from Newton's laws

$$F_t = ma_t.$$

Multiplying both sides of this relation by r, the distance of the particle from the axis of rotation, this relation becomes

$$F_t r = mra_t,$$

and writing a similar relation for each particle of the body, or system, we have by summation

$$\sum F_t r = \sum mra_t.$$

Let α denote the angular acceleration about the axis. Then $a_t = r\alpha$, and consequently this expression becomes

$$\sum F_t r = \sum m r^2 \alpha.$$

Since α is the same for each particle of the body, or system, this may be written

$$\sum F_t r = \alpha \sum m r^2.$$

The left member here denotes the total moment of the external applied forces with respect to the axis of rotation. Denoting this turning moment, or **torque**, as it is called, by T, the above relation becomes

$$T = \alpha \sum m r^2.$$

Comparing this equation for rotation with the fundamental equation for translation, namely, $F = ma$, it is evident that they are analogous, the difference being that in one angular acceleration and moment of force take the place of linear acceleration and force in the other. Hence to complete the analogy between the two expressions, Euler introduced the term "moment of inertia" for the quantity $\sum mr^2$, which replaces mass or inertia in the corresponding equation. Denoting this quantity by I, the initial letter of the word "inertia," that is, putting

$$I = \sum m r^2,$$

the fundamental equation for rotation about a fixed axis becomes simply

$$T = I\alpha.$$

Properly speaking, the quantity $I = \sum mr^2$ should be called the second moment of mass, although the term "moment of inertia" is more commonly used. The numerical value of I evidently depends on the shape and material of the body. For a solid of given shape it is a geometrical property which may be found by integration or its equivalent, as explained in Art. 116 and 117. For solids of irregular shape, the value of I may also be found experimentally by observing the angular acceleration produced in them by a torque of given amount, as explained in Art. 126.

It may also be mentioned in this connection that the central acceleration $r\omega^2$, where ω denotes the angular velocity about the axis of rotation, gives rise to an inertia force on any particle of mass m of amount (Art. 22)

$$F_n = mr\omega^2.$$

Summing up these forces for the entire body, the centrifugal force exerted on the axis is found to be

$$\sum F_n = \sum mr\omega^2 = \omega^2 \sum mr.$$

It will be shown in Art. 64, however, that

$$\sum mr = Mr_0,$$

where M denotes the mass of the entire body, or system, and r_0 is the distance of its center of mass from the axis of rotation. Hence

$$\sum F_n = \omega^2 Mr_0,$$

and since this force acts in the direction of r_0, if a body is to rotate rapidly in bearings, its center of mass should lie in the axis of rotation, that is, we should make $r_0 = 0$; otherwise the bearings will be subjected to periodically varying forces. This condition is obtained in practice by placing the axis in a horizontal position and adding or removing weight until the body is in equilibrium in any position, as explained in Art. 154. At the same time, even if this condition is fulfilled, there will be a centrifugal couple, also tending to produce stress in the bearings, unless the axis of rotation is what is known as a principal axis of inertia of the body (Art. 174). This condition cannot be attained by statical means, but only by experiments made on the body while in rotation, and is therefore not ordinarily provided for in machinery.

PROBLEMS

149. A small balance wheel has an axle $2\frac{1}{4}$ in. in diameter around which a cord is wrapped. It is found that a weight of 5 lb. hung on this cord is just sufficient to overcome friction, and that when the weight is 25 lb., it descends 5 ft. in 3 sec. Find the moment of inertia of the balance wheel.

150. A wheel 6 ft. in diameter has a moment of inertia of 600 lb. ft.², and is turning at the rate of 50 r. p. m. What opposing force applied tangentially to the rim will bring it to rest in one minute?

51. Energy of Rotation. — In the case of pure rotation the kinetic nergy of motion is expressed most simply in terms of the angular elocity. To obtain an expression for the energy of rotation of a gid body, or system of particles, constrained to rotate about a xed axis, consider any particle of the body of mass m, and let r enote its distance from the axis of rotation. Also let ds denote he linear distance traversed by the particle in the time dt, and $d\theta$ he angle which this arc subtends at the axis. The work done n the particle in causing it to describe the arc ds is then equal to s change in kinetic energy. That is, if F denotes the tangential omponent of the force acting on the particle, then, from Art. 44,

$$\int_{s_0}^{s} F ds = \tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0^2,$$

, since $ds = r d\theta$ and $v = r\omega$, where ω denotes the angular veocity of the body or system about the axis of rotation, this ecomes

$$\int_{\theta_0}^{\theta} F r d\theta = \tfrac{1}{2} mr^2\omega^2 - \tfrac{1}{2} mr^2\omega_0^2.$$

umming up for the entire body or system of particles we have, herefore,

$$\sum \int_{\theta_0}^{\theta} F r d\theta = \sum \tfrac{1}{2} mr^2\omega^2 - \sum \tfrac{1}{2} mr^2\omega_0^2,$$

hich may be written

$$\int_{\theta_0}^{\theta} \left(\sum F r \right) d\theta = \tfrac{1}{2} \omega^2 \sum mr^2 - \tfrac{1}{2} \omega_0^2 \sum mr^2.$$

he quantity $\sum mr^2$, however, is the moment of inertia I of the ody with respect to the axis of rotation, while the quantity $\sum Fr$ the total torque T of the external applied forces. This expreson may therefore be written

$$\int_{\theta_0}^{\theta} T d\theta = \tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\omega_0^2.$$

he quantity $\frac{1}{2} I\omega^2$ is called the kinetic energy of rotation, and is milar in form to the linear kinetic energy $\frac{1}{2} mv^2$, and is of the me dimensions as the latter. The relation just obtained, then, xpresses the fact that the work done by the torque T in produc-

ing the angular displacement θ is equal to the change in the kinetic energy of rotation of the body. If the torque T is constant during the motion, this relation becomes simply

$$T\theta = \tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\omega_0^2.$$

The same result may also be obtained by starting with the fundamental equation for the rotation of a rigid body about a fixed axis; namely, $T = I\alpha$. Writing this relation in the form $T = I\frac{d\omega}{dt}$, and integrating with respect to the angular displacement θ, we have

$$\int_{\theta_0}^{\theta} T d\theta = \int_{\theta_0}^{\theta} I \frac{d\omega}{dt} d\theta;$$

or, since $\frac{d\theta}{dt} = \omega$, this becomes

$$\int_{\theta_0}^{\theta} T d\theta = \int_{\omega_0}^{\omega} I\omega d\omega = \tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\omega_0^2,$$

which is identical with the result previously obtained.

The case most commonly met in practice is that in which the axis of rotation is a principal gravity axis; as, for example, in the case of an ordinary flywheel. Under this condition there is no tendency for the axis to alter its position, and the relation just obtained completely describes the motion.

In Art. 72 it will be shown that a solid body, or system of particles, moves as though all the forces applied to its various parts were applied at the center of mass to a single particle of mass equal to the mass of the entire body. This principle of the motion of the center of mass reduces the problem of determining the motion of the body, or system of particles, to that of finding the motion of a single particle, together with that of finding the relative motion of the body with respect to its center of mass. Thus the total kinetic energy of a body of mass M may be written

$$\text{kinetic energy} = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2,$$

where v denotes the linear velocity of the center of mass of the body, I the moment of inertia of the body with respect to an axis through its center of mass (gravity axis), and ω its angular velocity with respect to this axis. The center of mass is, in

general, the only point for which such a decomposition of the kinetic energy is possible.

If we introduce a quantity k, defined as

$$k = \frac{\Sigma mr^2}{\Sigma m},$$

so that k^2 is the mean value of r^2 averaged throughout the entire body, then

$$I = Mk^2,$$

where M denotes the mass, $\sum m$, of the entire body, and k is called the **radius of gyration**. The expression for the total energy of motion of a rigid body may then be written

$$\text{K. E.} = \tfrac{1}{2}\, Mv^2 + \tfrac{1}{2}\, I\omega^2 = \tfrac{1}{2}\, M(v^2 + k^2\omega^2).$$

To illustrate the use of this relation, consider a thin steel hoop of mass M and radius r, rolling along a horizontal plane, and let v denote the linear speed of the center of the hoop and ω its angular speed about this center. Assuming the entire mass of the hoop to be concentrated in the rim, we have $k = r$, and therefore since we also have $v = r\omega$, the expression for the total energy of motion becomes in this case

$$\text{K. E.} = \tfrac{1}{2}\, M(v^2 + v^2) = Mv^2.$$

That is to say, the total kinetic energy is twice the energy of translation of the center of mass.

PROBLEMS

151. A flywheel of a shearing machine has 150,000 ft.-lb. of energy stored in it when its speed is 250 r. p. m. How much energy does it part with during a reduction of speed to 200 r. p. m.?

152. In the preceding problem, if 82 per cent of the energy given out is imparted to the shears during a stroke of 2 in., what is the average force on a blade of the shears?

153. A car weighs 20,000 lb. and has 8 wheels, each weighing [illegible] lb. Considering the weight of the wheels as concentrated in the rim, compare the energy of rotation of each wheel with its energy of translation.

154. Show from the preceding problem that in computing the energy of a moving car it is sufficient to consider it as a single material particle of weight equal to the weight of the body plus twice the weight of the wheels, and compute only the energy of translation of this particle.

155. In order to control an engine against variations in external work, it is necessary to call upon the flywheel for 50,000 ft.-lb., with an allowable change in speed from 170 to 150 r.p.m. Assuming that the diameter of the wheel is 20 ft., compute its weight.

NOTE. — For a well-designed flywheel the radius of gyration k lies between $0.37\ D$ and $0.43\ D$, where D denotes the diameter. A good practical rule to follow is to assume $k = 0.4\ D$.

52. Angular Impulse and Momentum. — The principle of impulse and momentum deduced in Art. 47 for a motion of translation may be extended to include a motion of rotation as well. To prove this, consider a rigid body, or system of particles rigidly connected, constrained to rotate about a fixed axis. Let m denote the mass of any particle of the body, or system, at a distance r from the axis of rotation, and ω its angular velocity about this axis at any instant. Then the linear velocity of the particle at the instant considered is $r\omega$, and its tangential acceleration is $r\dfrac{d\omega}{dt}$. Consequently if F_t denotes the tangential component of the force acting on m, we have by Newton's laws

$$F_t = mr\frac{d\omega}{dt},$$

and the moment equation for the particle of mass m with respect to the axis of rotation is

$$F_t r = mr^2\frac{d\omega}{dt}.$$

Summing up for the entire body, or system of particles, we have therefore

$$\sum F_t r = \sum mr^2\frac{d\omega}{dt}.$$

The quantity $\sum F_t r$, however, is equal to the torque T of the external applied forces with respect to the axis of rotation, whereas $\sum mr^2 = I$, the moment of inertia of the body, or system of particles, with respect to this axis. Hence the equation of motion becomes $T = I\dfrac{d\omega}{dt}$, or simply

$$Tdt = Id\omega.$$

Integrating between any two time limits t and t_0, at which the angular velocities are ω and ω_0, respectively, we have

$$\int_{t_0}^{t} Tdt = \int_{\omega_0}^{\omega} Id\omega = I\omega - I\omega_0.$$

Consequently the impulse, or time integral of the applied torque, is equal to the change in angular momentum, which proves the theorem.

If the torque T is constant throughout the interval considered, this relation becomes

$$Tt = I\omega - I\omega_0.$$

The derivation just given may be shortened by starting from the equation for the rotation of a rigid body about a fixed axis; namely, $T = I\alpha$ (Art. 50). Writing this relation in the form $T = I\dfrac{d\omega}{dt}$, and integrating with respect to the time, we have as before

$$\int_{t_0}^{t} T dt = \int_{\omega_0}^{\omega} I d\omega = I\omega - I\omega_0.$$

When the action of the torque is instantaneous, it is called an impulsive couple. As explained in Art. 47, an impulsive couple must therefore be considered as an infinite torque exerted for an infinitesimal length of time. In this case the impulse, $\int_{t_0}^{t} T dt$, cannot be calculated directly, but is measured by the finite change in angular momentum produced; namely, $I\omega - I\omega_0$.

PROBLEMS

156. An engine in starting exerts on the crank shaft for one minute a constant torque of 1000 ft.-lb. There is also a resisting moment acting of [illegible] ft.-lb. The flywheel has a radius of gyration of 5 ft. and weighs 2000 lb. What speed will the engine attain in one minute?

NOTE. — Let I denote the moment of inertia of the flywheel, W its weight, and k its radius of gyration. Then $I = \dfrac{W}{g}k^2$.

157. What torque is required to bring a flywheel having a moment of inertia of 1800 lb. ft.^{2} from rest to a speed of 50 r. p. m. in one minute?

53. Conservation of Angular Momentum. — Since linear momentum, mv, is the product of two factors, one of which is a scalar and the other a vector, their product is also a vector. The moment of this vector with respect to any point O is then also a vector, and may be represented by a line through O perpendicular to the plane of O and mv (Fig. 84). This vector is called the **moment of momentum** with respect to O, or the **angular momentum**.

In Art. 48 it was proved that the total linear momentum of any system of particles or rigid bodies is unchanged by any action between them; that is to say, as long as no external forces are introduced into the system, the resultant vector momentum is constant. It can also be shown that the moment of this resultant with respect to any fixed axis is constant; that is to say, when no external forces act upon the system, its angular momentum with respect to a fixed axis is conservative.

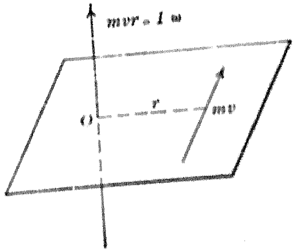


FIG. 84

To prove the principle of the conservation of angular momentum, let x, y, z denote the rectangular components of any particle of mass m. Then, as in Art. 48, if X, Y, Z denote the rectangular components of the force acting on this particle, its motion is determined by the equations

$$X = m\frac{d^2x}{dt^2}, \quad Y = m\frac{d^2y}{dt^2}, \quad Z = m\frac{d^2z}{dt^2}.$$

From Fig. 85, however, the sum of the moments with respect to the X-axis of these components, that is, the torque about the X-axis, is

$$yZ - zY.$$

Substituting the values of the force components Y and Z in terms of the component accelerations, this becomes

$$m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right).$$

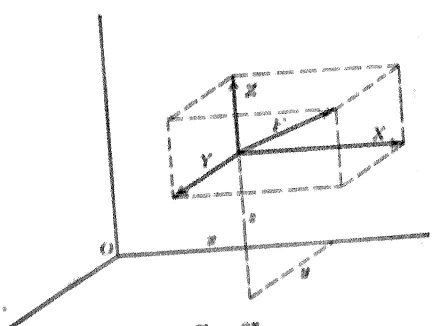


FIG. 85

Since the rectangular components of the linear momentum of the particle are $m\frac{dx}{dt}, m\frac{dy}{dt}, m\frac{dz}{dt}$, the sum of the moments of these components about the X-axis, that is, the X component of the angular momentum, is

$$m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

Differentiating this expression with respect to the time, the result is

$$\frac{d}{dt}\left[m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)\right]=m\left(\frac{dy}{dt}\frac{dz}{dt}+y\frac{d^2z}{dt^2}-\frac{dz}{dt}\frac{dy}{dt}-z\frac{d^2y}{dt^2}\right)$$

$$=m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=yZ-zY.$$

This result is true for each particle of a rigid body, or system of particles rigidly connected. By summation throughout the entire body, or system of particles, we have therefore

$$\frac{d}{dt}\sum m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)=\sum(yZ-zY).$$

The quantity $m\sum\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)$, which is the sum of the angular momenta of the separate particles with respect to the X-axis, is the X component of the angular momentum of the body, or system of particles, and will be denoted by H_x. The internal forces contribute nothing to this expression since they occur in pairs, equal in amount and opposite in sign. The right member of the equation, $\sum(yZ-zY)$, represents the total torque about the X-axis exerted by the external applied forces. Denoting it by T_x, the above relation becomes, therefore,

$$\frac{dH_x}{dt}=T_x.$$

Similarly

$$\frac{dH_y}{dt}=T_y,$$

$$\frac{dH_z}{dt}=T_z.$$

Consequently the rate of change of the angular momentum of a rigid body, or system of particles, about any axis is equal to the torque about the same axis of the external applied forces.

When no external forces act on the body, we have $T_x=T_y=T_z=0$, and consequently

$$\frac{dH_x}{dt}=0,\quad \frac{dH_y}{dt}=0,\quad \frac{dH_z}{dt}=0,$$

or, integrating with respect to the time,

$$H_x = \text{constant}, \quad H_y = \text{constant}, \quad H_z = \text{constant}.$$

Therefore if no external forces act upon the body, its angular momentum remains constant. This is the principle of the conservation of angular momentum.

PROBLEMS

158. A shaft and pulleys keyed to the shaft are rotating at 200 r. p. m., the total moment of inertia of pulleys and shaft being [illegible] lb. ft.² A loose pulley, or idler, at rest on the shaft is suddenly connected up with the fixed pulleys by means of a clutch. If the moment of inertia of the loose pulley is [illegible] lb. ft.², find the angular velocity with which the whole will start to revolve.

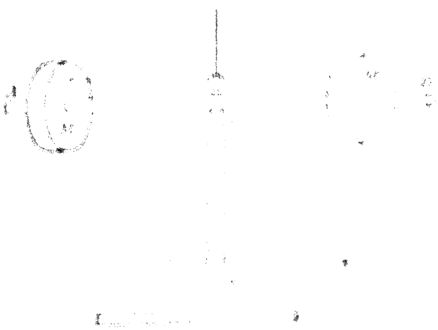


Fig. [illegible]

159. Two weights, each of mass m, are free to slide along a horizontal rod AB (Fig. [illegible]). If AB is revolving about the vertical axis CD with an angular velocity ω and the weights are suddenly pulled in toward the axis by means of the strings passing through the ring at C, what will be the new angular velocity?

54. Summary of Results. All the fundamental formulas of dynamics have been deduced in this chapter from Newton's laws, as expressed by $F = ma$. These results may be summarized as follows:

	Translation	Rotation
Fundamental equation	$F = ma$	$T = I\alpha$
Work and energy	$\int F\,ds = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$	$\int T\,d\theta = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$
Impulse and momentum	$\int F\,dt = mv - mv_0$	$\int T\,dt = I\omega - I\omega_0$

If the force or torque is constant throughout the motion, and the initial velocity is zero, these results simplify into the following:

Work and energy	$Fs = \frac{1}{2}mv^2$	$T\theta = \frac{1}{2}I\omega^2$
Impulse and momentum	$Ft = mv$	$Tt = I\omega$

Since these formulas simply represent the space and time integrals of the fundamental equation $F = ma$, or its equivalent, $T = I\alpha$, it is evident that they express only what is contained in Newton's laws. Moreover, no new principle will be introduced in what follows, although the relations given above will be extended so as to express certain more general relations, chief of which will be the principle of the conservation of energy, d'Alembert's principle, the principle of virtual work and the principle of least work.

The entire system of mechanics here presented is thus based entirely on Newton's laws. Since the subject originated in this way, it is undoubtedly the best for an elementary treatment. Newton's axioms, however, are not the only possible logical foundation for the subject. An entirely different set of axioms or postulates may be laid down as fundamental, and the subject approached from a different standpoint. Nothing new is obtained in this way, however, as the results must be susceptible of experimental verification, and hence the fundamental assumptions, in whatever form they may be stated, must be substantially equivalent to those given by Newton. Those interested in the philosophical aspect of mechanics are referred to *The Principles of Mechanics*, by Heinrich Hertz; and *The Science of Mechanics*, by Ernest Mach.

55. Fundamental and Derived Units. — To measure a quantity it is necessary to have a quantity of the same kind to serve as a unit or standard for comparison. The quantity is then measured by finding how many times it contains the unit of measurement chosen as a standard; that is, by finding the number which expresses the ratio of the given quantity to the chosen unit.

The quantities employed in mechanics, however, are not all independent, as shown by the relations between them, derived above and also in the preceding chapter. For example, velocity, length, and time are connected by the relation $s = vt$. If, then,

two of these three quantities, say length and time, are chosen as fundamental units, the third, namely, velocity, can be expressed in terms of these two, and is thereby determined as the ratio of the units of length and time.

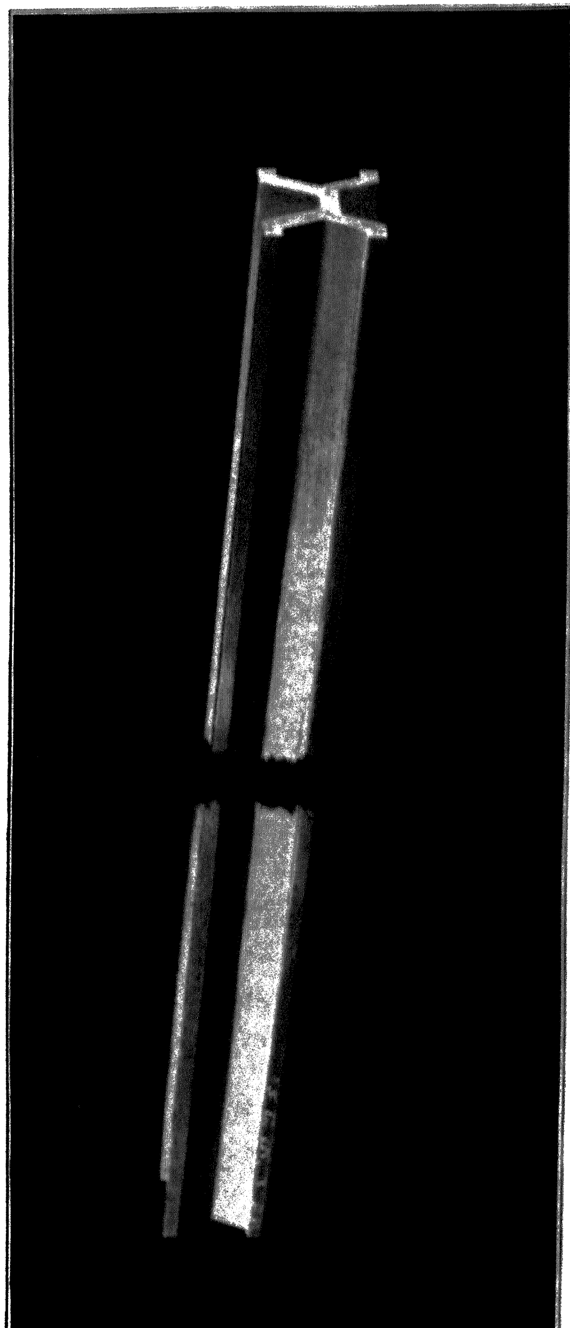
Since the dynamical equations listed on p. 112 are equivalent to only one independent relation, $F = ma$, and the kinematical equations on p. 29 are equivalent to two independent relations, there are in all only three independent relations between the six quantities F, m, v, a, s, t. Hence three of these quantities may be chosen arbitrarily as fundamental units. No more than three can be chosen arbitrarily, since the units of measurement for the remaining three quantities are connected with the three chosen by the three relations mentioned.

The three units chosen arbitrarily are called **fundamental units**, and all the other units used in mechanics are called **derived units**.

56. Unit of Time. As kinematics is the oldest branch of mechanics, the units of length and time were the two units first chosen as fundamental. The unit of time has a natural standard or prototype in the solar day and its subdivisions, of which $\frac{1}{86400}$, or the second, has been universally accepted as the unit of time from the most remote past.

57. Unit of Length. — The unit of length has been variously assumed, and not until 1875 was there any universally accepted standard. In that year, by the action of the seventeen principal governments of the world, an international standard of length was adopted. This consists of the distance between two lines at 0° C. on a platinum-iridium bar deposited at the International Bureau of Weights and Measures, Sevres, France.* This bar is called the International Prototype Standard, and copies of it, called national prototype standards, were deposited in 1889 with the Bureau of Weights and Measures, Washington, D.C., the Board of Trade, London, and with the various other governments of the world (Fig. 87).

* The meter was originally supposed to be $\frac{1}{10^7}$ of the arc of a meridian extending from the equator to the pole, but later determinations have shown that the measurements from which this value was deduced were inexact. It is probable that some other unit, such as the wave length of homogeneous light, may ultimately be chosen as the unit of length.



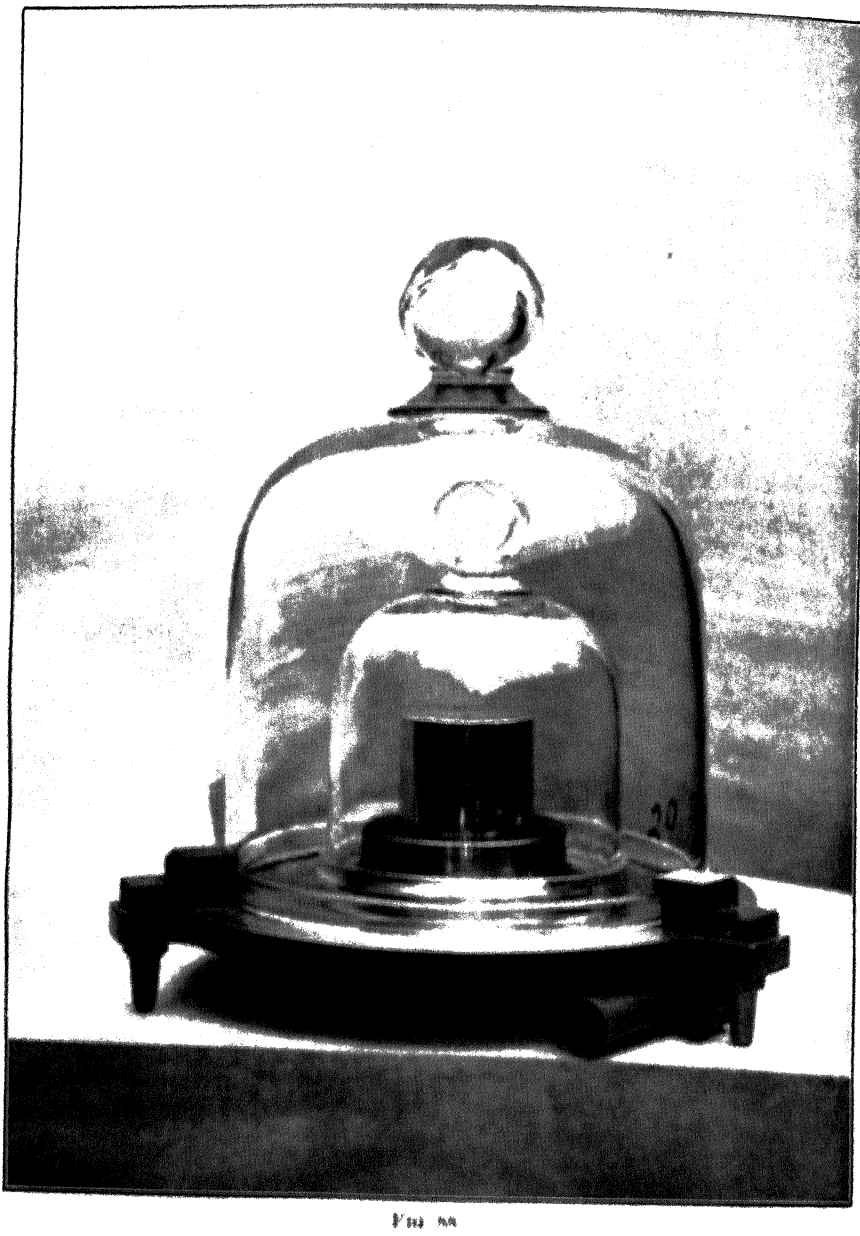


FIG. 88

The English yard and foot still continue to be used, but are expressed in terms of the meter. In the United States the yard was defined by Act of Congress, July 28, 1866, as

$$1 \text{ U. S. yard} = \tfrac{3600}{3937} \text{ m.},$$

and similarly the British imperial yard is defined by law as

$$1 \text{ British imperial yard} = \tfrac{3600}{3937.079} \text{ m.}$$

Two other units of length in common use are the statute or land mile, and the nautical or sea mile. The first is equivalent to 5280 feet and is our standard of itinerary measure, adopted from the English, who in turn adopted it from the Romans. A Roman military pace, by which distances were measured, was approximately five feet long, and a thousand of these paces was called in Latin a "mille." The English mile is therefore a purely arbitrary measure, legalized by a statute passed during the reign of Queen Elizabeth.

A nautical mile is supposed to be equal to the length of the arc of a great circle of the earth subtended by an angle of one minute at its center. Since the earth is not spherical, the nautical mile so defined would vary with the latitude. For this reason the English admiralty has adopted 6080 ft. as the length of a nautical mile, which corresponds to the length of one minute of arc of a great circle in latitude 48°; while the United States Coast Survey has adopted 6080.27 ft. for the nautical mile, which is equivalent to the length of one minute of arc on a great circle of a sphere whose surface is equal to the surface of the earth. One nautical mile is therefore equivalent to 1.1515 statute miles, or one statute mile is equivalent to 0.869 nautical mile.

A knot is the nautical unit of speed, and is defined as one nautical mile per hour. Thus if a vessel is said to have a speed of 20 knots, it is equivalent to saying that it has a speed of 20 nautical miles per hour.

58. Unit of Weight. — With the inception of dynamics, a third unit, in addition to length and time, became necessary. This was first chosen as the unit of force, and was defined as the weight of one cubic decimeter, or liter, of water at its temperature of greatest density, 3.9° C. This was later replaced by what was intended

to be an equal weight of platinum, called the kilogram. This was constructed in the form of a cylinder, with diameter and height each equal to 39 mm., under direction of the French Academy at the same time as the standard meter, and was likewise deposited among the Archives of France. Copies of this standard were also deposited with the governments of this and other countries at the same time as the National Prototype Meters, and are now the legal standards in these countries (Fig. 88). By Act of Congress of 1866 the pound was defined in terms of the kilogram as

$$1 \text{ pound avoirdupois} = \frac{1}{2.2046} \text{ kilogram.}$$

Similarly, the British imperial pound is defined by law as

$$1 \text{ Imperial pound} = \frac{1}{2.2046212} \text{ kilogram.}$$

59. Unit of Mass. — When the National Prototype Kilograms were distributed to the various nations, it was found that they differed slightly in weight from the International French Prototype, owing to change in latitude. For this reason the *mass* of these platinum cylinders was chosen as the fundamental unit instead of their weight, as it was thus possible to establish an absolute international standard.

In this way there originated two systems of units, called the **gravitation system** and the **absolute system**, respectively. The gravitation system is that used in engineering, commerce, and the arts, and is based on length, time, and force (*i.e.* weight) as fundamental units. The absolute system is used in theoretical investigations, and is based on length, time, and mass, as fundamental units.

60. Gravitation System. — In the gravitation system two sets of fundamental units are in common use, one based on the international or metric units, the kilogram, meter, and second, and the other based on their British equivalents, the foot, pound, and second. Hence there are also two sets of derived units in the gravitation system. Thus the unit of velocity is either 1 cm./sec. or 1 ft./sec.; the unit of work is either the kilogrammeter or the foot-pound, etc.

61. Absolute System. — The absolute system of units was originated by Gauss and Weber, who used the millimeter as unit of length, the milligram as unit of mass, and the second as unit of time. Since 1873, however, the centimeter, gram, and second have been universally accepted as the fundamental units, and for this reason the absolute system is often called the C. G. S. system. It is now used almost exclusively in theoretical investigations in astronomy, physics, electricity, and mechanics.

Special names have been devised for the derived units in the C. G. S. system to distinguish them from the corresponding units in the gravitation system. Thus the unit of velocity is called the *kine;* the unit of acceleration the *spoud;* the unit of force the *dyne;* the unit of work or energy the *erg;* the unit of impulse or momentum the *bole*, etc.

In practical work, especially in electricity, it is desirable to use larger units than those just named. Thus there has originated a practical system, based on multiples of the C. G. S. units, namely, the meter, kilogram, and second, and called the M. K. S. system. Many of the derived units in this system are named after men who were prominent in mechanical pursuits. Thus the unit of work or energy is called the *joule;* the unit of power the *watt*, etc.

62. Dimensions of Units. — All kinematical and dynamical formulas are simply algebraic statements of certain relations which exist between the various quantities involved, and consequently between the units in terms of which these quantities are measured. When stated so as to express simply the relations existing between the derived and fundamental units, such equations are called **dimensional formulas.** By the dimensions of any term of a formula is meant the power to which the various units occurring in this term are raised. This is in accordance with ordinary usage; thus it is customary to speak of an area as the square of a length, of a volume as the cube of a length, etc., the square or cube being the dimensions of the area or volume; that is to say, the power to which the fundamental unit, length, occurs.

The chief characteristic of a dimensional equation is that each term must be homogeneous in the fundamental units involved. That is to say, when all the units involved are expressed in terms

of the fundamental units, each term of the resulting expression must contain the same powers of these units. Otherwise the given expression is inconsistent with fundamental mechanical principles.

In the absolute system the fundamental units are length, time, and mass, the symbols for which are L, T, M. The dimensional equations which express the dimensions of the various derived units may then be deduced from the kinematical and dynamical relations previously given.

For velocity, or speed, the kinematical relation is $v = \frac{l}{t}$, from which the dimensional relation is unit velocity $= \frac{\text{unit distance}}{\text{unit time}}$, or, symbolically,

$$V = \frac{L}{T} = LT^{-1}.$$

For acceleration, the kinematical relation is $a = \frac{v}{t}$, and hence the dimensional relation is $A = \frac{V}{T}$, or, since $V = LT^{-1}$, this becomes

$$A = \frac{LT^{-1}}{T},$$

or

$$A = LT^{-2}.$$

Force is defined by the dynamical relation $F = ma$, whence from the above its dimensions are

$$F = MLT^{-2}.$$

Using this result, the dimensions of an impulse Ft are $MLT^{-2} \cdot T$,

or

$$\text{Impulse} = MLT^{-1}.$$

Similarly for momentum mv, since $V = LT^{-1}$, its dimensions are

$$\text{Momentum} = MLT^{-1}.$$

The dimension of impulse and momentum are thus identical, which of course agrees with the dynamical relation between them. Since the expression for work is $W = Fs$, its dimensions are

$$W = ML^2T^{-2}.$$

Similarly for kinetic energy $E = \frac{1}{2} mv^2$, the dimensions are

$$E = ML^2T^{-2},$$

which also agrees with the dynamical relation between work and energy. In other words, the equations which express the principles of work and energy, and of impulse and momentum, are homogeneous in the fundamental units.

Power is defined as unit power $= \dfrac{\text{unit work}}{\text{unit time}}$, and hence its dimensions are $\dfrac{ML^2T^{-2}}{T}$, or

$$P = ML^2T^{-3}.$$

Torque, or moment of force, is the product of force and distance. Hence unit torque, or moment of force, = unit force × unit distance, and the corresponding dimensional equation is

$$\text{Torque} = ML^2T^{-2}.$$

Thus torque is of the same dimensions as work or energy, although as a dynamical quantity it is entirely distinct.

Stress is defined as unit stress $= \dfrac{\text{unit force}}{\text{unit area}}$, and hence its dimensions are

$$\text{Stress} = ML^{-1}T^{-2}.$$

Density is defined as unit density $= \dfrac{\text{unit mass}}{\text{unit volume}}$, and consequently its dimensions are

$$\text{Density} = ML^{-3}.$$

To obtain the dimensions of the unit of heat, it is necessary to introduce a fourth fundamental unit; namely, the unit of temperature. This unit is also entirely arbitrary, and as in the case of the unit of length is measured by a double standard, called the Fahrenheit and Centigrade scales, respectively. The latter has the advantage of being metric, and is based on the amount of heat required to raise water from the temperature of melting ice to the boiling point. The unit of temperature on this scale is usually obtained by noting the expansion of a mercury column for this range of temperature, and dividing the interval so obtained into one hundred equal parts called degrees.

Let D, then, denote the unit of temperature on any scale. Then the unit of heat, called the **calorie**, is defined as the amount of heat required to raise unit mass through unit temperature (one gram through one degree Centigrade). Consequently the dimensions of a heat unit H are

$$H = DM.$$

It was found by Joule that to produce a unit of heat required a certain number of units of work. This number is called **Joule's equivalent**, and will be denoted by J. Thus

$$1 \text{ unit heat} = J \text{ units work.}$$

To obtain the dimensions of J, suppose it is found by experiment that H units of heat $= W$ units of work. Then since 1 unit heat $= J$ units work, the relation becomes $HJ = W$, and consequently the dimensions of J are $\dfrac{L^2MT^{-2}}{DM}$, or

$$J = L^2D^{-1}T^{-2}.$$

The most important application of dimensional equations is to the change of units from one system to another, say from the British to the metric, or the reverse. The use of dimensional equations for this purpose is illustrated in the following problems.

PROBLEMS

160. Express the value of Joule's equivalent in the British system.

SOLUTION. The mean value of Joule's equivalent in the metric system is

$$J = 4.2 \times 10^7 \text{ ergs},$$

the unit of temperature being 1° C. Let J_F denote the required value of J in foot-pounds, referred to the Fahrenheit scale. Then the dimensional equation becomes

$$32.2\, J_F L^2 D^{-1} T^{-2} = 4.2 \times 10^7 L_1^2 D_1^{-1} T^{-2}.$$

Since 100 cm. $= \dfrac{39.37}{12}$ ft. and 5° C. $=$ 9° F., we have $L_1 = \dfrac{39.37}{1200} L$ and $D_1 = \dfrac{9}{5} D$. Consequently

$$J_F = \frac{4.2 \times 10^7}{32.2} \left(\frac{39.37}{1200}\right)^2 \times \frac{5}{9} = 780.1 \text{ ft.-lb.}$$

161. Find the number of watts in a horsepower.

SOLUTION. Let x denote the number of ergs per second in one horsepower. Then since 1 h. p. = 550 ft.-lb./sec. the dimensional equation is, in

this case, $xL^2MT^{-3} = 32.2 \times 550\, L_1^2 M_1 T^{-3}$.

Since $\qquad 1000 \text{ g.} = 2.2046 \text{ lb.}, \; M_1 = \dfrac{1000}{2.2046} M$, and therefore

$$x = 32.2 \times 550 \left(\frac{1200}{39.37}\right)^2 \left(\frac{1000}{2.2046}\right) = 746 \times 10^7 \text{ ergs/sec.},$$

or $\qquad 1 \text{ h.p.} = 746$ watts.

162. Determine the relation between mass and weight in the British and metric systems.

Solution. In the gravitation system, where unit force is defined as the attraction between the earth and unit mass, the unit of acceleration is g, the average value of which is 32.2 ft./sec.², or 981 cm./sec.², whereas in the absolute system it is 1 ft./sec.² or 1 cm./sec.². Hence the relation between the gravitation and absolute units of acceleration is, in the British system, $A = 32.2\, A_1$, and in the metric system, $A = 981\, A_1$. Therefore from the dynamical equation $F = ma$, the dimensional formula is $F = MA = MA_1$, whence

$$1 \text{ lb. weight} = 1 \text{ lb. mass} \times 32.2 \text{ ft./sec.}^2,$$

and

$$1 \text{ g. weight} = 1 \text{ g. mass} \times 981 \text{ cm./sec.}^2.$$

In the British system a force which gives an acceleration of 1 ft./sec.² to a mass of 1 lb. is called a poundal, and similarly in the metric system, a force which gives an acceleration of 1 cm./sec.² to a mass of 1 g. is called a dyne. Hence the above relations may also be written

$$1 \text{ lb. weight} = 32.2 \text{ poundals},$$
$$1 \text{ g. weight} = 981 \text{ dynes}.$$

163. In the British system $g = 32.2$ ft./sec.². Find its value in the metric system in cm./sec.².

Solution. Let x denote the required value of g in the metric system. Then

$$32.2\, LT^{-2} = xL_1T^{-2},$$

whence

$$x = 32.2 \frac{1200}{39.37} = 981.5.$$

164. Find the relation between the poundal and the dyne.

Solution. Let x equal the number of dynes in a poundal. Then

$$MLT^{-2} = xM_1L_1T^{-2},$$

whence

$$x = \frac{ML}{M_1L_1} = \frac{M}{M_1} \cdot \frac{120000}{3937}.$$

Since 1 cu. cm. of water weighs 1 g., and 1 cu. ft. of water weighs 1000 oz.,

$$\frac{M}{M_1} = \frac{1 \text{ lb.}}{1 \text{ g.}} = \frac{1}{\dfrac{1000}{16}\left(\dfrac{39.37}{1200}\right)^3}.$$

Consequently

$$x = \frac{16}{1000}\left(\frac{120000}{3937}\right)^4 = 13{,}810.$$

CHAPTER III

STATICS

63. Center of Gravity. — Consider a system of n particles, lying in the same plane and rigidly connected. The weights of these particles, w_1, w_2 . . . w_n, constitute a system of forces directed toward the center of the earth. This, however, is relatively at an infinite distance as compared with the distances between the particles, and hence their weights may be regarded as a system of parallel forces.

The total weight W of all the particles is

$$W = w_1 + w_2 + \cdots + w_n = \sum w;$$

that is, W is the vector resultant of the n parallel vectors w_1, w_2 . . . w_n. The position of W may be determined by means of the theorem of moments deduced in Art. 24; namely, that the sum of the moments of w_1, w_2 . . . w_n about any point is equal to the moment of their resultant W about this point. To apply this theorem, let O be any point in the plane of the particles, and let x_1, x_2 . . . x_n denote the perpendicular distances of the vectors w_1, w_2 . . . w_n from O; or, what amounts to the same thing, the perpendicular distances of P_1, P_2 . . . P_n from a vertical through O (Fig. 89). Then, if W represents the weight of the entire system and x_0 its distance from O, by the principle of moments

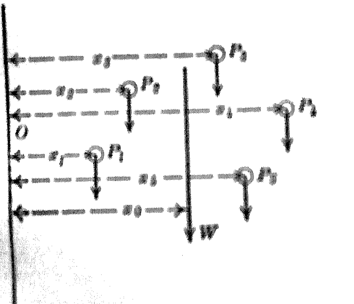


Fig. 89

$$Wx_0 = w_1x_1 + w_2x_2 + \cdots + w_nx_n = \sum wx,$$

whence

$$x_0 = \frac{\Sigma wx}{W}.$$

Since $W = \sum w$, this may also be written

$$x_0 = \frac{\Sigma wx}{\Sigma w}.$$

Thus the magnitude of the resultant W is determined by the relation $W = \sum w$, and its line of action by $x_0 = \frac{\Sigma wx}{\Sigma w}$. There still remains to be found the particular point at which the resultant W is applied. For this purpose let the system of particles be turned through any angle, and determine the line of action of W in this new position by the same method as above. The two lines of action so determined will then intersect in a point, called the **center of gravity**. From the manner in which this point is found it is evident that if the entire weight of all the particles was concentrated at the center of gravity, this single force would be equivalent to the given system of forces, no matter in what direction the system is turned.

If the particles $P_1, P_2 \cdots P_n$ do not all lie in the same plane, a reference *plane* must be drawn through O instead of a reference line. In this case the equation $x_0 = \frac{\Sigma wx}{\Sigma w}$ determines the position of a plane in which the vector W must lie. The intersection of three such planes, corresponding to different positions of the system of particles, will then determine a point which is the required center of gravity.

64. Center of Inertia, or Mass. — Let $m_1, m_2 \cdots m_n$ denote the masses of the n particles just considered. Then

$$w_1 = m_1 g, \quad w_2 = m_2 g, \cdots w_n = m_n g,$$

and $W = Mg$, where M denotes the total mass of the entire system. Hence the relations determining the center of gravity may be written

$$Mg = \sum mg, \quad x_0 = \frac{\Sigma mgx}{\Sigma mg},$$

or, since g is constant,

$$M = \sum m, \quad x_0 = \frac{\Sigma mx}{\Sigma m}.$$

The point determined from these relations by taking the system of particles in two or more positions is called the **center of inertia**, or **center of mass**. Since these relations are equivalent to those given in the preceding article, it is evident that the center of mass is identical with the center of gravity.

65. Center of Gravity of Solids. — A solid body may be considered as made up of an infinite number of heavy particles or material points, rigidly connected. Its center of gravity, or center of mass, is then obtained precisely as above, except that the summation must now be extended over an infinite number of infinitesimal particles and is therefore replaced by an integration. Hence if dw denotes the weight of an infinitesimal particle and dm its mass, the equations for determining the center of gravity of a solid are

$$W = \int dw, \quad x_0 = \frac{\int x\,dw}{\int dw};$$

and those for determining its center of mass are

$$M = \int dm, \quad x_0 = \frac{\int x\,dm}{\int dm}.$$

PROBLEMS

165. Find the center of gravity of a pyramid of height h.

SOLUTION. Cut the pyramid into slices parallel to the base, as shown in Fig. 90. Then if b denotes the **area** of the section, the volume of the slice is $b\,dx$, or, since by geometry $\frac{b}{x^2} = \frac{B}{h^2}$, volume $= \frac{B}{h^2} x^2 dx$, and consequently the moment of the slice about the apex is $\frac{B}{h^2} x^3 dx$. Hence

$$x_0 = \frac{\int_0^h \frac{B}{h^2} x^3 dx}{\int_0^h \frac{B}{h^2} x^2 dx} = \frac{3}{4} h.$$

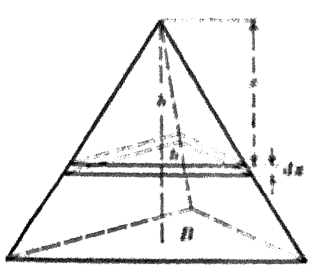


FIG. 90

166. Determine the center of gravity of a right circular cone of height h.

167. Find the center of gravity of a homogeneous hemisphere of radius r.

168. Find the center of gravity of a homogeneous paraboloid of revolution of height h.

169. Find the center of gravity of a hemisphere in which the density varies directly as the distance from the base.

66. Centroid. — It is sometimes desirable to determine the point designated as the center of gravity or center of mass without reference to either the mass or weight of the body, but simply with respect to its geometric form.

For a solid body let dV denote an element of volume and δ its density. Then since mass is jointly proportional to volume and density,

$$dm = \delta dV.$$

Consequently the formulas of the preceding article may be written

$$\delta V = \int \delta dv, \quad x_0 = \frac{\int x\delta dv}{\int \delta dv};$$

or, since δ is constant throughout,

$$V = \int dv, \quad x_0 = \frac{\int x dv}{\int dv}.$$

Since the point previously called the center of gravity, or center of mass, is now determined simply from the geometric form of the body, it is designated by the special name **centroid.**

The centroid of an area or line may also be determined from these equations, although, properly speaking, neither has a center of gravity or center of mass, since an area or length has neither weight nor mass. For an area the centroid is determined by the equations

$$A = \int da, \qquad x_0 = \frac{\int x da}{\int da}.$$

where da denotes an element of area, and A the total area. For a line, the equations for determining the centroid are

$$L = \int dl, \quad x_0 = \frac{\int x\,dl}{\int dl},$$

where dl denotes an element of length, and L the total length of the line or curve.

PROBLEMS

170. Determine the centroid of a circular arc or wire in terms of the angle it subtends at the center.

SOLUTION. Let r denote the radius of the arc, and 2β the central angle (Fig. 91). Then in polar coördinates, $x = r\cos\theta$, $ds = r\,d\theta$, and the moment of ds about O is $ds \cdot r\cos\theta = r^2\cos\theta\,d\theta$. Hence $x_0 = \dfrac{2\int_0^\beta r^2\cos\theta\,d\theta}{2r\beta} = \dfrac{r\sin\beta}{\beta}$.

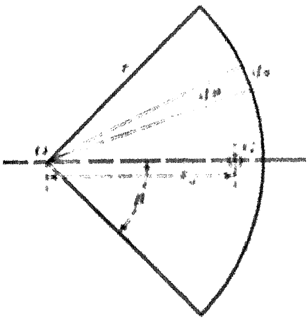


FIG. 91

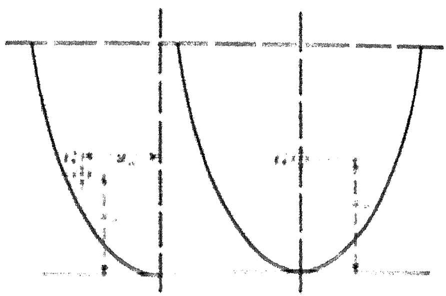


FIG. 92

171. From the preceding problem, determine the centroid of a semicircular arc or wire.

172. From Prob. 170, determine the centroid of a quadrantal arc or wire.

173. Determine the centroid of a parabolic arc or wire for the two cases shown in Fig. 92.

174. Find the centroid of a triangle.

175. Find the centroid of a semicircle.

176. Find the centroid of a circular quadrant.

177. Determine the centroid of a circular sector in terms of the length of the arc and the chord subtended by it.

178. Find the centroid of a parabolic segment for the two cases shown in Fig. 93.

179. Find the centroid of a semi-ellipse.

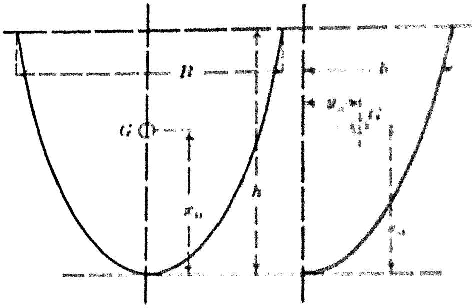


Fig. 93

67. Axis of Symmetry. A plane or axis of symmetry of a homogeneous solid, or of an area or line, always contains the center of gravity or centroid; for to an element of the figure on one side of the plane or axis of symmetry there always corresponds an equal element, symmetrically placed, on the opposite side of the plane or axis, as illustrated in Fig. 94. Hence the moments of these two elements about the plane or axis are equal in amount but of opposite sign, and consequently their sum is zero. Since the moment of each pair of elements is identically zero, the total moment is also zero, and hence the center of gravity or centroid lies in the plane or axis of symmetry (Fig. 95).

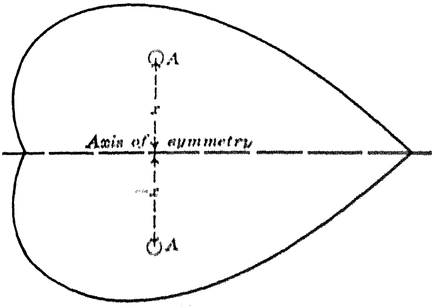


Fig. 94

When a figure has two or more planes or axes of symmetry, their intersection completely determines the centroid or center of gravity (Fig. 96).

68. Composite Figures. To determine the centroid of a figure composed of several parts, the centroid of each part may first be determined separately. Then assuming that each figure (mass, weight, volume, area,

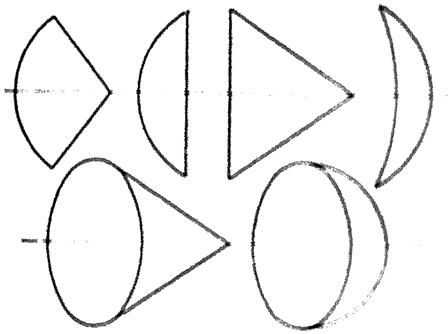


Fig. 95

or length) is concentrated at its centroid, the centroid of the entire figure may be found by applying the theorem of moments, as expressed by the formulas in Art. 66.

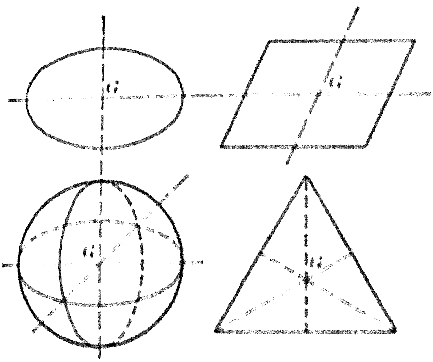


FIG. 96

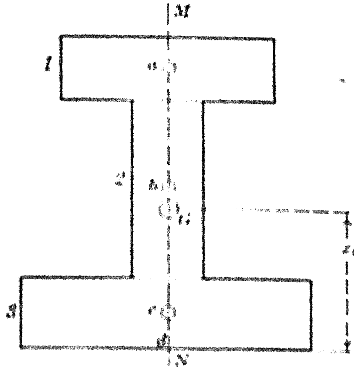


FIG. 97

To illustrate this method, let it be required to find the centroid of the cross section of an *I* beam, as shown in Fig. 97. Since the figure has an axis of symmetry *MN*, the centroid must lie somewhere on this line. To find its position, divide up the figure into three rectangles, as shown by the dotted lines. The centroids of these rectangles, considered separately, are at their centers *a*, *b*, *c*. Hence, denoting the areas of these rectangles by *A*, *B*, *C*, respectively, the position of the centroid of the entire figure is given by

$$x_0 = \frac{A \cdot \overline{ad} + B \cdot \overline{bd} + C \cdot \overline{cd}}{A + B + C}.$$

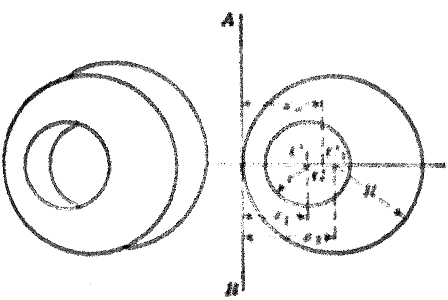


FIG. 98

This method also applies to solids. For example, consider the cam, or eccentric, shown in Fig. 98, which consists of a circular disk of uniform thickness with a circular hole cut eccentrically. In this case, a line joining the centers of the circles and lying

midway between the plane faces is an axis of symmetry, and hence the center of gravity lies somewhere on it. Its position on this line may be found by determining the centroid of the plane face of the figure. Taking moments about any line, say a tangent perpendicular to the line of centers,

$$x_0 = \frac{\pi R^2 x_2 - \pi r^2 x_1}{\pi R^2 - \pi r^2},$$

or, since $x_2 = R$ and $x_1 = R - e$, where e denotes the eccentricity or distance between centers,

$$x_0 = \frac{R^3 - r^2(R - e)}{R^2 - r^2}.$$

PROBLEMS

180. Find the center of gravity of the frustum of a right circular cone.

181. Find the center of gravity of the frustum of a right pyramid with square base.

69. Experimental Determination of Center of Gravity. — When a body is supported at one point, as, for example, when suspended by a string, then in order that the body may be in equilibrium, the supporting force and that due to the weight of the body must be equal in amount, opposite in direction, and act along the same line. Hence the point of support and the center of gravity must lie in the same vertical. This gives a means for determining the center of gravity experimentally. For if a body is supported at one point and a vertical drawn through this point, the center of gravity must lie somewhere in this vertical. If then it is supported at some other point, not in this line, and a new vertical drawn, the center of gravity must also lie in this second vertical. The intersection of these two lines, therefore, determines the center of gravity.

To find the centroid of a plane area, it may first be traced on a piece of cardboard of uniform thickness, and cut out along the boundary. Then balance it on a knife edge, and mark the line of support. By balancing it in two different positions the center of gravity of the lamina (and consequently the centroid of the plane area) will thus be determined by the intersection of the two lines so determined.

70. Graphical Determination of the Centroid. When a plane figure has an irregular outline, the centroid cannot be determined analytically because the equations of the boundary are unknown. In this case, however, the equations for determining the centroid may be applied graphically, as follows.

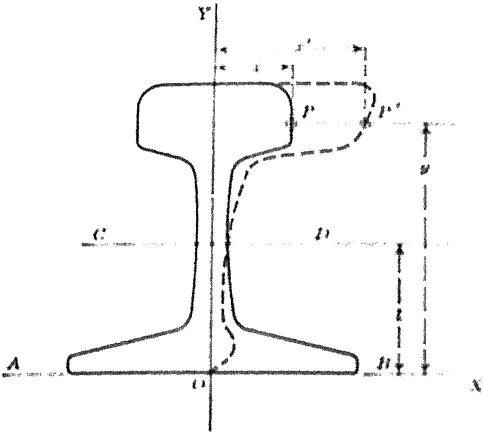


FIG. 99

Consider an irregular section, such as the rail section shown in Fig. 99. Since the figure is symmetrical, the centroid must lie on the axis of symmetry, which will be taken for the Y-axis. Take any line AB as a base, and draw any other parallel line CD at an arbitrary distance l from it. Then transform the boundary of the section by means of the relation

$$x' = x\frac{y}{l};$$

that is to say, the x coordinate of every point P of the boundary is multiplied by its distance y from the base line and divided by the constant l. By taking a sufficient number of points P of the boundary, and for each x computing the corresponding x', a new boundary is obtained, shown by the dotted line in Fig. 99.

From the relation previously deduced, the y coordinate of the centroid is given by the relation

$$y_0 = \frac{\int y da}{\int da}$$

Taking a horizontal strip of width dy for an element of area, $da = xdy$, and hence the expression for the centroid becomes

$$y_0 = \frac{\int xydy}{\int xdy}$$

Since from the equation for transformation of the boundary $xy = lx'$, this may be written

$$y_0 = \frac{l\int x'dy}{\int xdy}.$$

The integral in the numerator, however, is the area of the transformed section, say A', while the denominator is the area A of the original section. Hence

$$y_0 = \frac{lA'}{A}.$$

By measuring these two areas A and A' by means of a planimeter, or otherwise, the centroid may thus be easily determined.

71. Application to finding Areas and Volumes.

I. The centroid of a plane curve is determined by the relation

$$x_0 l = \int x dl.$$

Now suppose that this curve is revolved about an axis in its plane, and let A denote the area of the surface of revolution so generated. Then the area of an infinitesimal zone of this surface of radius x is the circumference of its base $2\pi x$, multiplied by its length dl. Hence the area of the entire surface is

$$A = 2\pi \int x dl.$$

Eliminating the integral between this expression and the one for the centroid, the result is

$$A = 2\pi x_0 l,$$

which may be expressed by the following theorem: *

Area of Surface of Revolution. *The area of a surface of revolution, generated by revolving a plane curve about an axis in its plane, is equal to the product of the length of the curve by the circumference of the circle described by its centroid.*

* Due to Pappus, who lived and taught at Alexandria about the end of the third century.

II. The centroid of a plane area is given by the relation

$$x_0 A = \int x\,da.$$

The volume of the solid of revolution generated by revolving this area about an axis in its plane is given by

$$V = 2\pi \int x\,da.$$

Hence, eliminating the integral between these two expressions, the result is

$$V = 2\pi x_0 A,$$

which may be stated as follows:*

Volume of Solid of Revolution. *The volume of a solid of revolution, generated by revolving a plane area about an axis in its plane, is equal to the product of this area by the circumference of the circle described by its centroid.*

III. Consider a truncated right prism, or cylinder, and let A denote the area of the upper base, and B the area of the lower base (Fig. 100). Let the lower base be divided up into n equal rectangles of area db, so that $db = \frac{B}{n}$, and pass planes through its edges perpendicular to the base, so as to divide the figure into small rectangular prisms. Then the upper base will also be divided into n equal parts, each of area $da = \frac{A}{n}$. Let x denote the height of one of these elementary prisms; then its volume will be $x\,db$. Hence increasing the number of divisions, n, indefinitely, the volume of the entire prism or cylinder becomes

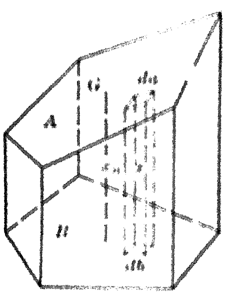


FIG. 100

$$V = \lim_{n \to \infty} \sum x\,db;$$

or, since $db = da\frac{B}{A}$, this may be written,

$$V = \lim_{n \to \infty} \sum \frac{B}{A} x\,da = \frac{B}{A}\int x\,da.$$

* Due to Guldinus, born 1577, died 1643; taught in Jesuit College at Gratz.

Therefore, since $x_0 = \dfrac{\int x da}{A}$, the result is, finally,

$$V = B x_0.$$

Expressed in words,

The volume of a truncated right prism or cylinder is equal to the area of the right section multiplied by the distance of the center of gravity of the truncated base from this right section.

PROBLEMS

182. Find the area and volume of a sphere by means of theorems I and II.

183. Find the area and volume of a paraboloid of revolution by means of theorems I and II.

184. Find the centroid of a semicircular arc from theorem I, being given the area of a sphere.

185. Find the volume of an ellipsoid of revolution by theorem II.

186. Find the surface and volume of an anchor ring, or torus, generated by revolving a circle about an external axis.

187. Find the volume of the ring generated by revolving an ellipse about an external axis.

72. Properties of the Center of Gravity. — Consider a rigid body of mass M, and let $\bar{x}, \bar{y}, \bar{z}$ denote the coördinates of its center of gravity, and $\bar{v}_x, \bar{v}_y, \bar{v}_z$ the components of the velocity of this center. Then

$$\bar{v}_x = \frac{d\bar{x}}{dt}, \quad \bar{v}_y = \frac{d\bar{y}}{dt}, \quad \bar{v}_z = \frac{d\bar{z}}{dt},$$

and also from the definition of the center of gravity,

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma mz}{\Sigma m}.$$

Consequently

$$\bar{v}_x = \frac{d\bar{x}}{dt} = \frac{d}{dt}\left(\frac{\Sigma mx}{\Sigma m}\right) = \frac{\Sigma m\dfrac{dx}{dt}}{\Sigma m} = \frac{\Sigma mv_x}{M},$$

and similarly

$$\bar{v}_y = \frac{\Sigma mv_y}{M}, \quad \bar{v}_z = \frac{\Sigma mv_z}{M},$$

or, clearing of fractions,

$$M\bar{v}_x = \sum mv_x, \quad M\bar{v}_y = \sum mv_y, \quad M\bar{v}_z = \sum mv_z.$$

Since the right members of these equations represent the components of the total linear momentum of the body, the following theorem may be stated:

The linear momentum of a rigid body is equal to the mass of the body multiplied by the velocity of its center of gravity.

Now let X, Y, Z denote the components of the force acting upon any element of the body of mass m. Then by Newton's law, the motion of this particle is determined by the equations

$$X = \frac{d}{dt}(mv_x),\quad Y = \frac{d}{dt}(mv_y),\quad Z = \frac{d}{dt}(mv_z),$$

where v_x, v_y, v_z denote the components of the velocity of the particle. (Compare Art. 48, Chapter II.) Summing up for the entire body, we have therefore

$$\sum X = \frac{d}{dt}\left(\sum mv_x\right),\quad \sum Y = \frac{d}{dt}\left(\sum mv_y\right),\quad \sum Z = \frac{d}{dt}\left(\sum mv_z\right),$$

and replacing the quantity in each parenthesis by its value obtained above, namely, $\sum mv_x = M\bar{v}_x$, etc., we have

$$\sum X = \frac{d}{dt}(M\bar{v}_x) = M\frac{d\bar{v}_x}{dt},\quad \sum Y = M\frac{d\bar{v}_y}{dt},\quad \sum Z = M\frac{d\bar{v}_z}{dt}.$$

Therefore, since $\frac{d\bar{v}_x}{dt}$, $\frac{d\bar{v}_y}{dt}$, $\frac{d\bar{v}_z}{dt}$ represent the components of the acceleration of the center of gravity of the body, the following theorem may be stated:

The motion of the center of gravity of a rigid body is the same as though the entire mass of the body was concentrated at its center of gravity and all the external forces were applied at this point.

These two theorems greatly simplify the dynamics of rigid bodies, since they prove that for any motion of translation the mass may be assumed to be concentrated at the center of gravity, and the motion of this point alone be considered.

73. Composition and Resolution of Forces. — In Chapter I it was shown that the sum of two or more vectors, such as displacements, velocities, or accelerations, can be represented graphically by the

closing side of the vector triangle or polygon formed on these vectors as sides. This closing side of the vector polygon is called the resultant of the given system of vectors.

When a body of mass m receives an acceleration, it implies that it is acted upon by a force F, given by the fundamental relation $F = ma$. Therefore if a body receives several accelerations, say a_1, a_2, a_3, it must be acted upon by an equal number of forces F_1, F_2, F_3, each of which bears the ratio m to the corresponding acceleration; namely,

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3} = m.$$

Since the resultant of these vector accelerations may be found by means of a vector polygon, as explained in Chapter I, the corresponding forces may also be so combined, since they are respectively parallel and proportional to the corresponding accelerations and consequently the two polygons are similar (Fig. 101).

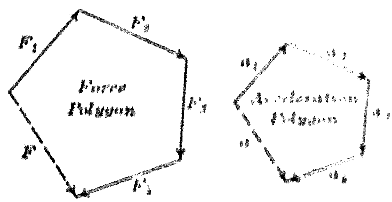


FIG. 101

By reversing this process, a force F may be resolved into any number of components, F_1, F_2, F_3, in arbitrary directions, the only condition being that the force F and its components shall form a closed figure.

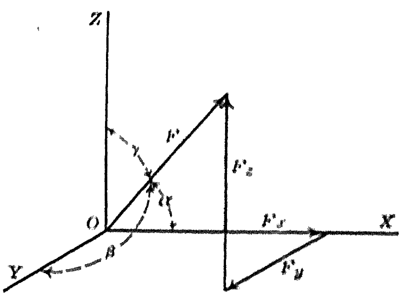


FIG. 102

It is frequently convenient to apply this principle by resolving a force into components parallel to a set of rectangular axes. In this case the required components are simply the projections of the given force on the axes (Fig. 102).

When a number of forces are to be considered, this is the most convenient method of procedure. Each force is resolved into rectangular components X, Y, Z, and the sums of these components taken, say,

$$F_x = \sum X; \quad F_y = \sum Y; \quad F_z = \sum Z.$$

The resultant F of the sums F_x, F_y, F_z is then the resultant of the entire given system of forces. Its numerical value is evidently

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}.$$

Also, if α, β, γ denote the angles the resultant F makes with the axes (Fig. 102), its direction is given by

$$\cos\alpha = \frac{F_x}{F}, \quad \cos\beta = \frac{F_y}{F}, \quad \cos\gamma = \frac{F_z}{F}.$$

PROBLEMS

188. Two equal and opposite parallel forces constitute what is called a **couple.** The moment of a couple is equal to the product of either force and the distance between them. Prove that the moment of a couple about any point whatever is constant.

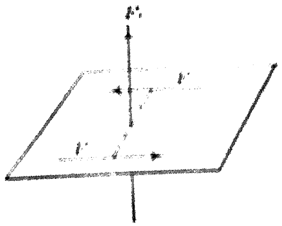


FIG. 103

189. A couple may be represented by a single vector, perpendicular to the plane of the couple, equal in length to the moment of the couple, and pointing in the direction in which a right-handed screw would advance if rotated in the direction indicated by the couple (Fig. 103). Show from this that a couple may be revolved through any angle without changing its value.

190. Prove that the sum of the moments of two forces about any point is equal to the moment of their resultant about this point (Varignon's theorem).

NOTE. — Take any point O for the given point, draw OA, and take a line AB through A perpendicular to OA for base line (Fig. 104). Then take moments about O, and make use of the relation that the sum of the projections of the two given forces on the base line is equal to the projection of the resultant on this line.

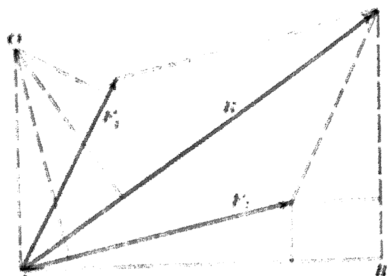


FIG. 104

191. Prove that any system of coplanar forces is equivalent to a single force and a couple.

HINT. — At any point O, introduce two equal and opposite forces for each of the given forces, so that one of these forces, together with the given force, will form a couple, leaving an unbalanced force, equal to the given force, at O. Having done this for each of the given forces, the couples so formed may be combined into a single couple, and the forces at O into a single resultant.

192. Show that when three coplanar forces are in equilibrium, they must meet in a point, and that each force is proportional to the sine of the angle between the other two (Lami's theorem).

193. What is the magnitude of the couple required to transfer a force of 10 lb. to a point 4 ft. from its line of action?

194. What is the resultant of a couple of 12 ft.-lb. and a single force of 4 lb.?

195. In moving a force to a point 5 ft. distant, a couple of 30 ft.-lb. was introduced. What was the magnitude of the force?

74. Conditions of Equilibrium. When a body acted upon by two or more forces is at rest or in uniform motion relative to any system of coördinate axes, it is said to be in equilibrium, and the forces acting on it are said to equilibrate. The condition that a body shall be in equilibrium is that its resultant acceleration shall be zero. From the relation $F = ma$, however, if the resultant acceleration a is zero, the resultant force F must also be zero. Hence the condition for equilibrium against translation is simply

$$F = 0;$$

or, if F is the resultant of a system of forces $F_1, F_2, \cdots F_n$, the condition becomes

$$\sum_1^n {}_r F_r = 0.$$

It is usually convenient to resolve each of the given forces into components X, Y, Z parallel to the coördinate axes before applying this condition, in which case the single condition for equilibrium just given breaks up into the three separate conditions

$$\sum X = 0, \quad \sum Y = 0, \quad \sum Z = 0.$$

These conditions apply only to the equilibrium of the center of gravity of a body. It is possible, however, for the center of gravity to be in equilibrium, and yet for the body to rotate about this point, as occurs, for example, in the case of a flywheel rotating about a fixed axis. Hence in order that a body may also be in equilibrium as regards rotation, a further condition is necessary; namely, that its angular acceleration α shall also be zero. From the fundamental relation $T = I\alpha$, if $\alpha = 0$, the turning

moment, or torque, T must also be zero. Hence the condition for equilibrium against rotation is

$$T = 0;$$

or, if T is the resultant of several moments $T_1, T_2, \cdots T_n$, this condition becomes

$$\sum_1^n {}_r T_r = 0.$$

These two conditions,

for equilibrium against translation $\qquad \sum F = 0,$

for equilibrium against rotation $\qquad \sum T = 0,$

constitute the basis of the entire subject of statics. Typical illustrations of their application are given in what follows.

PROBLEMS

196. The average turning moment exerted on the handle of a screw driver is 120 in.-lb. The screw has a square slot, but the point of the screw driver is beveled to an angle of 10° (Fig. 105). If the point of the screw driver is ¼ in. wide, find the vertical force tending to raise the screw driver out of the slot.

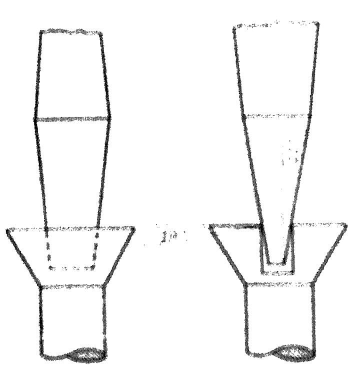


FIG. 105

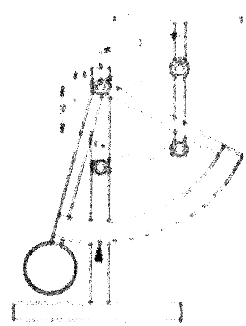


FIG. 106

197. In the letter scales shown in Fig. 106, the length of the parallel links is 3 in. and the distance of the center of gravity of the moving parts below the pivot O is 2 in. If the radius of the scale is 8 in. and the weight of the moving parts is 12 oz., find the distance between successive ounce graduations on the scale.

198. A scale is arranged as shown in Fig. 107. Determine the relation between the load and the weight P. (Quintenz Scales, Strassburg, 1821.)

SOLUTION. With the given dimensions we have by the principle of moments

$$R_1 = W\frac{x}{l},\quad F_1 = W\frac{l-x}{l},\quad F_2 = R_1\frac{d}{e+d},$$

and $$Pa = F_1 b + F_2 c,$$

whence $$Pa = Wb - W\frac{x}{l}\left(b - \frac{d}{e+d}c\right).$$

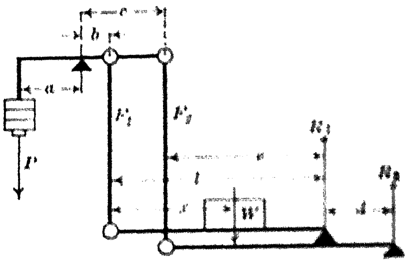


FIG. 107

Since this is independent of x, the position of the load on the platform does not affect the result.

Let the dimensions be so proportioned that $\frac{b}{c} = \frac{d}{e+d}$, and also $a = 10b$. Then $P = \frac{W}{10}$. A scale so arranged is called a decimal scale.

199. In the toggle joint press shown in Fig. 108, the length of the hand lever is $l_1 = 3\frac{1}{2}$ ft., and $l_2 = 4$ in. If the pull $P = 100$ lb., find the pressure between the jaws of the press when the toggle is inclined at 10° to the vertical.

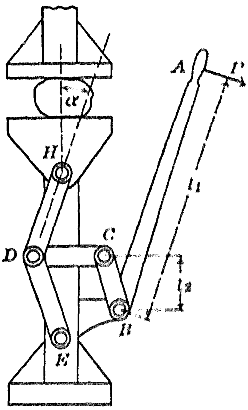


FIG. 108

200. Find the least horizontal force necessary to pull a wheel 30 in. in diameter and carrying a load of 500 lb., over an obstacle 4 in. high.

201. A steelyard weighs 6 lb. and has its center of gravity in the short arm at a distance of 1 in. from the fulcrum. The movable weight weighs 4 lb. Find the zero graduation, and the distance between successive pound graduations.

202. A differential screw consists of two screws, one inside the other. The outer screw works through a fixed block, and is turned by means of a lever. This screw is cored out and tapped for a smaller screw of less pitch which works through another block, free to move along the axis of the screw, but prevented from rotating. Find the mechanical advantage of such a differential screw if the lever arm is 3 ft. long, the outer screw has 8 threads to the inch, and the inner screw 10 threads to the inch.

203. The bed of a straight river makes an angle α with the horizontal. Taking a cross section perpendicular to the course of the river, the sides of the valley are inclined at an angle β to the horizontal. Find the angle which the tributaries of the river make with it.

204. In the various types of three-horse whiffletrees shown in Fig. 109, determine the ratios of the different lever arms so that each horse shall pull one third of the load.

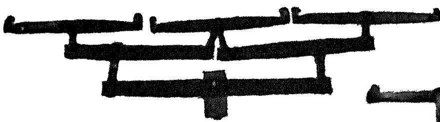


FIG. 109a

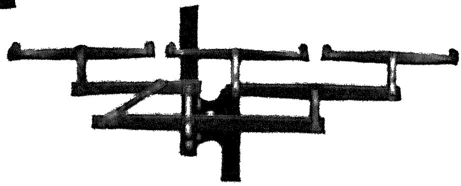


FIG. 109b

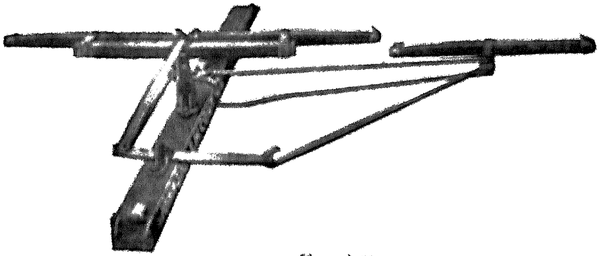


FIG. 109c

75. Equilibrium Polygon. As explained above, the resultant of any system of forces lying in the same plane may be found by means of a vector force polygon, the resultant being the closing side of the polygon formed on the given system of forces as adjacent sides. Although this construction gives the magnitude and direction of the resultant, it does not determine its position, or line of action. The most convenient way to determine the line of action of the resultant is to introduce into the given system two equal and opposite forces of arbitrary amount and direction, such as P' and P'' (Fig. 110). Since P' and P'' balance one another, they will not affect the equilibrium of the given system. To find the line of action of the resultant R, combine P' and P_1 into a resultant R_1 acting along $B'A'$, parallel to the corresponding ray OB of the force polygon. Prolong $A'B'$ until it intersects P_2 and then combine R_1 and P_2 into a resultant

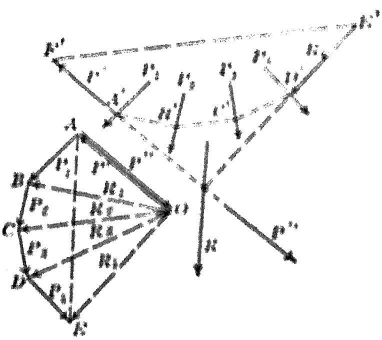


FIG. 110

R_2, acting along $C''B'$, parallel to the corresponding ray OC' of the force polygon, etc. Proceed in this way until the last partial resultant R_4 is obtained. Then the resultant of P'' and R_4 will give the line of action as well as the magnitude of the resultant of the original system P_1, P_2, P_3, P_4. The closed figure $A'B'C'D'E'F'$ obtained in this way is called **an equilibrium polygon.**

For a system of parallel forces the equilibrium polygon is constructed in the same manner as above, the only difference being that in this case the force polygon becomes a straight line (Fig. 111).

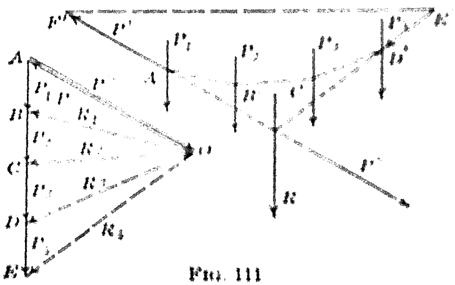


FIG. 111

Since P' and P'' are entirely arbitrary both in magnitude and direction, the point O, called the **pole**, may be chosen anywhere in the plane. Therefore, in constructing an equilibrium polygon corresponding to any given system of forces, the force polygon $ABCDE$ (Fig. 110 and 111) is first drawn, then any convenient point O is chosen and joined to the vertices A, B, C, D, E of the force polygon, and finally the equilibrium polygon is constructed by drawing its sides parallel to the rays OA, OB, OC, etc., of the force diagram. Since the position of the pole O is entirely arbitrary, there are an infinite number of equilibrium polygons corresponding to any given set of forces. The position and magnitude of the resultant R, however, is independent of the choice of the pole, and will be the same no matter where O is placed.

For a system of concurrent forces (*i.e.* forces which all pass through the same point) the closing of the force polygon is the necessary and sufficient condition for equilibrium. If, however, the forces are not concurrent, or are parallel, this condition is necessary but not sufficient, for in this case the given system of forces may be equivalent to a couple, the effect of which would be to produce rotation. To assure equilibrium against rotation, therefore, it is also necessary that the equilibrium polygon shall close.

The graphical and analytical conditions for equilibrium are then as follows:

CONDITIONS OF EQUILIBRIUM

	Analytical	Graphical
Translation	$\sum F = 0$	Force polygon closes
Rotation	$\sum T = 0$	Equilibrium polygon closes

76. Principle of Virtual Work. — If a system consists of a single free particle (*e.g.* center of gravity of rigid body, Art. 72) in order for it to be in equilibrium, the resultant of all the forces applied to it must be zero. That is to say, if X, Y, Z denote the components of this resultant, then for equilibrium

$$X = Y = Z = 0.$$

It is convenient, however, to consider this condition for equilibrium in a somewhat different form. Thus suppose that the particle is displaced slightly from its position of equilibrium, and let δx, δy, δz denote the components of this displacement. Then if the particle is in equilibrium, the condition $X = Y = Z = 0$ may be written

$$X\delta x + Y\delta y + Z\delta z = 0;$$

that is to say, the work done in the given small displacement is zero. Conversely, if the work done in any small arbitrary displacement is zero, then in order that the relation $X\delta x + Y\delta y + Z\delta z = 0$ shall be satisfied for any arbitrary values of δx, δy, δz, each term must vanish separately; that is, $X = 0$, $Y = 0$, $Z = 0$, in which case the body is in equilibrium.

In the case of a system of particles, if each particle is independent of the others, the condition just obtained may be applied to each one separately. If, however, the particles are connected in any way, the motion of each one is constrained by that of the others. That is, any displacement of one or more particles of the system causes a corresponding displacement of all the others.

By a **virtual displacement** of the system is meant a displacement consistent with the constraints imposed on the system. For example, in the case of the bent lever shown in Fig. 114 the point

of application of each of the forces is constrained to move in the arc of a circle about the center O. If, then, the lever arms are of length l_1 and l_2 and the lever is turned through an angle α, the ends of the lever are constrained to move through arcs $l_1\alpha$ and $l_2\alpha$, respectively.

The **Principle of Virtual Work**, which is a generalization of the condition obtained above for the equilibrium of a single free particle, may be stated as follows:

If any system of particles (or rigid bodies), each acted on by any number of forces, is in equilibrium, the sum total of the work done by all the forces in any small virtual displacement of the system is zero.*

The truth of this principle was first perceived by Stevinus at the close of the sixteenth century from his investigations on the equilibrium of systems of pulleys. Thus in Fig. 137 and 138, if the forces F and W are in equilibrium and F is displaced through a distance a, W will be constrained to move through a distance b such that, neglecting friction, $Fa = Wb$, or $Fa - Wb = 0$; that is to say, the total work done on the system in any small virtual displacement is zero.

Galileo extended this principle somewhat by applying it to an inclined plane. Thus suppose that a body of weight P rests on a plane of inclination α and is supported by a hanging weight W attached to P by a string passing over a pulley at the top of the plane. Then in case the bodies are in equilibrium, if W descends a distance h, P will be raised vertically a distance $k = h \sin \alpha$ such that $Wh = Pk$, or $Wh - Pk = 0$. In this relation Galileo perceived that the equilibrium of the bodies depended not only on their weights, but also on their relative movement towards and from the center of the earth.

The universal applicability of the principle of virtual work was discovered by John Bernoulli in 1717, and a formal proof of the principle was given by Lagrange in his *Analytical Mechanics*. For a critical discussion of the principle see Mach, *Science of Mechanics*, 2d ed., 1902, Chapter IV.

The following problems illustrate the practical application of the principle in a number of simple cases. These problems, of course,

* That is, vanishes to at least the first order of magnitude.

can be solved by other means which may appear simpler. The value of a general principle such as that of virtual work consists in the fact that it involves economy of thought. For example, if a machine was entirely inclosed so as to be invisible except for the points of application of two forces P and Q, and if a displacement of P through a distance a caused a corresponding displacement of Q through a distance b, we would know from this principle that for equilibrium of the entire system, $Pa = Qb$, irrespective of how the machine is constructed. The principle of virtual work thus affords a general method which may be applied to all problems in static equilibrium and therefore obviates the necessity of investigating each particular case separately.

PROBLEMS

205. A body of weight W is supported by two strings, one of which is horizontal, and the other inclined at an angle α to the horizontal. Find by the principle of virtual work the tension in the strings.

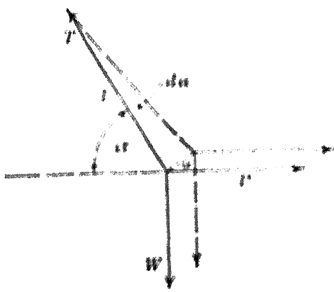


FIG. 112

SOLUTION. Assume a small (virtual) displacement of the system, as shown in Fig. 112. Then if the length of the string is l, the work done by the various forces is as follows:

$$\text{Work of } W = W \times l d\alpha \cos \alpha,$$
$$\text{Work of } P = P \times l d\alpha \sin \alpha,$$
$$\text{Work of } T = T d\alpha \times l d\alpha.$$

Hence $P l d\alpha \sin \alpha + T d\alpha l d\alpha = W l d\alpha \cos \alpha$,

or, neglecting infinitesimals of the second order, and dividing through by $l d\alpha$,

$$P = W \cot \alpha.$$

206. A movable pulley is supported by two cords equally inclined to the vertical. By the Principle of Virtual Work deduce the relation between the tension in the cord and the weight supported by the pulley.

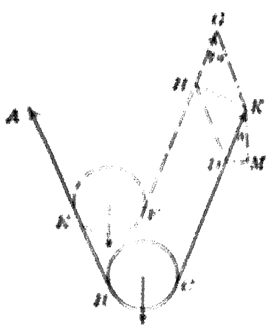


FIG. 113

SOLUTION. Let the pulley be displaced slightly from its position of equilibrium, as shown in Fig. 113. Then since the length of the cord is unchanged, $ABCD = AEFG$. Subtract this from the equality $AB + BC + CK = AB + EF + FG$.

Then $KD = BE = KG$. Hence if P is the pull in the cord,

$$\begin{aligned}\text{Virtual work of left-hand cord} &= P \cdot KG,\\ \text{Virtual work of right-hand cord} &= P \cdot GH,\\ \text{Virtual work of weight} &= W \cdot KM.\end{aligned}$$

Therefore $$P \times KG + P \times GH = W \times KM,$$

or, since $$GH = KG \cos 2\alpha \text{ and } KM = KG \cos \alpha,$$

$$P(1 + \cos 2\alpha) = W \cos \alpha.$$

Making use of the relation $1 + \cos 2\alpha = 2\cos^2 \alpha$, this becomes

$$W = 2P \cos \alpha.$$

207. Obtain the relation between the forces acting upon a bent lever (Fig. 114) by means of the Principle of Virtual Work.

208. A weight W rests upon a plane of inclination θ to the horizontal, and is supported by a force P, making an angle α with the plane. By means of the Principle of Virtual Work determine the relation between P and W.

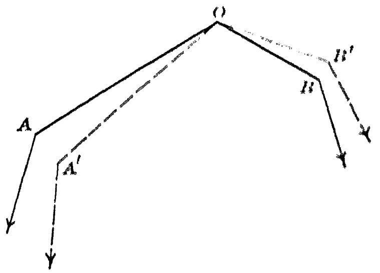


FIG. 114

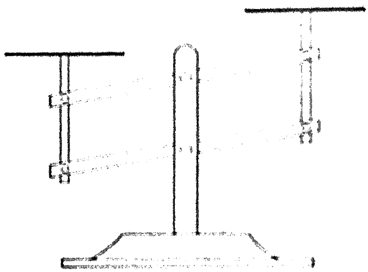


FIG. 115

209. In the Roberval's balance, shown in Fig. 115, show by means of the Principle of Virtual Work that the position of the weights in the scalepans does not affect the result of the weighing.

210. A smooth screw of pitch p supports a weight W by means of a force F applied at a distance d from its axis. Find the relation between F and W by the Principle of Virtual Work.

77. Stable, Unstable, and Neutral Equilibrium. — Let P denote the potential energy possessed by a body or material particle in any given position, and X, Y, Z, the components of a force by which it is displaced slightly from this position. Let the components of this displacement be denoted by δx, δy, δz. Then, since the total work done on the body must equal its change in potential energy, say δP, we have

$$\delta P = X\,\delta x + Y\,\delta y + Z\,\delta z.$$

Since the potential energy P is a function of the coördinates x, y, z, δP may be expressed in terms of its partial derivatives with respect to x, y, z, by means of the calculus formula for a total differential; namely,

$$\delta P = \frac{\partial P}{\partial x}\delta x + \frac{\partial P}{\partial y}\delta y + \frac{\partial P}{\partial z}\delta z.$$

Equating these two expressions for δP, we have

$$X\,\delta x + Y\,\delta y + Z\,\delta z = \frac{\partial P}{\partial x}\delta x + \frac{\partial P}{\partial y}\delta y + \frac{\partial P}{\partial z}\delta z.$$

In order for this expression to be independent of the direction of the displacement, the coefficients of δx, δy, δz on opposite sides of the equation must be equal; that is,

$$X = \frac{\partial P}{\partial x},\quad Y = \frac{\partial P}{\partial y},\quad Z = \frac{\partial P}{\partial z}.$$

Consequently, the force acting on the body in any given direction is equal to the partial derivative of the potential energy taken in the same direction.

For a body to be in equilibrium, the resultant force acting on it must be zero; that is, $X = 0$, $Y = 0$, $Z = 0$, and consequently

$$\frac{\partial P}{\partial x} = 0,\quad \frac{\partial P}{\partial y} = 0,\quad \frac{\partial P}{\partial z} = 0.$$

But this is the calculus condition that P shall be a maximum or a minimum. Consequently, the condition that a body shall be in equilibrium is that its potential energy shall be either a maximum or a minimum.

For a system of particles or bodies, the potential energy of the system is a function of all the coordinates of all the particles comprised in the system. If, then, the system is in equilibrium, each particle must be in equilibrium, and consequently the components of the force acting on each particle must be zero. Therefore the partial derivatives of the potential energy of the system with respect to the coordinates of each particle must be zero. But these are the conditions that the potential energy of the entire system shall be a maximum or a minimum. Hence, in a position

of equilibrium, the potential energy of any system of particles or bodies is either a maximum or a minimum.

Now consider a single material particle or body, or a system of such particles or bodies, acted on by any number of conservative forces, and suppose that the system is in equilibrium and at rest in the given position. Suppose, also, that it is subjected to constraints so that it can move from this position along only one path; as, for example, in the case of a ball resting in a groove. As mentioned above, the potential energy of any system is a function of all the coördinates of the system. Thus, if the potential energy P has a certain value, say P_A, when the system is in a given configuration A, it will in general have a different value P_B in any other configuration B; that is, when displaced slightly from its original position. Now let s denote any coördinate which measures how far the configuration of the system has been displaced from the original position A. Thus, for the ball resting in a groove, s may denote how far it has rolled along the groove from its initial position, etc. Then, expanding the potential energy P by Maclaurin's theorem in terms of its value when the system is in the configuration A, we have

$$P = P_A + s\left(\frac{\partial P}{\partial s}\right)_A + \frac{s^2}{2}\left(\frac{\partial^2 P}{\partial s^2}\right)_A + \cdots .$$

Assuming that A is a position of equilibrium for the system, we have from the preceding $\left(\frac{\partial P}{\partial s}\right)_A = 0$. Therefore neglecting terms of order higher than the second since s is assumed to be small, the above expression becomes

$$P - P_A = \frac{s^2}{2}\left(\frac{\partial^2 P}{\partial s^2}\right)_A .$$

When the system is displaced slightly from A, since this new position is in general not one of equilibrium, it cannot remain at rest. When it begins to move, however, it must gain kinetic energy and therefore lose an equal amount of potential energy since the forces acting on the system are assumed to be conservative. We have then three cases to consider according as $\left(\frac{\partial^2 P}{\partial s^2}\right)_A$ is positive, negative, or zero.

CASE I. If $\left(\frac{\partial^2 P}{\partial s^2}\right)_1$ is positive, then $P - P_1$ must remain positive throughout the motion; that is, we must have $P > P_1$, because the other factor $\frac{s^2}{2}$ is essentially positive. Since P is decreasing, s must therefore also decrease. Hence when displaced from its position of equilibrium, the system moves back toward this position. The equilibrium in this case is said to be **stable.** Hence *for stable equilibrium the potential energy is a minimum.*

CASE II. If $\left(\frac{\partial^2 P}{\partial s^2}\right)_1$ is negative, then by the same reasoning as above $P - P_1$ must remain negative throughout the motion; that is, $P < P_1$. Therefore since P is decreasing, s must increase, and consequently when the system is displaced from its position of equilibrium and then released, it moves away from this position. The equilibrium in this case is called **unstable.** Hence *for unstable equilibrium the potential energy is a maximum.*

CASE III. If $\left(\frac{\partial^2 P}{\partial s^2}\right)_1 = 0$, then $P = P_1$; that is to say, the potential energy is not changed by the displacement, and consequently there is no tendency to move in either direction. The equilibrium in this case is called **neutral.** Hence *for neutral equilibrium the potential energy is constant.*

As a simple example of the different conditions of equilibrium, consider a heavy sphere resting on a level table. If the center of gravity of the sphere coincides with its center of figure, it may be placed in any position on the table and will show no tendency to move. It is then in neutral equilibrium. Its potential energy in this case is its weight multiplied by the distance of its center of gravity above a fixed horizontal plane of reference, and since neither factor is changed by the motion, its potential energy remains constant.

If the sphere is not homogeneous, so that its center of gravity does not coincide with its center of figure, there are two positions in which it will be in equilibrium; namely, when its center of gravity is directly above or directly below its geometric center.

In the first case the potential energy is evidently a maximum, since the distance of the center of gravity above a horizontal

plane of reference is the greatest possible. In this case, if the sphere is displaced slightly, the motion will continue, and it is therefore in unstable equilibrium.

If the center of gravity is directly below the geometric center, it is in its lowest possible position, and the potential energy is therefore a minimum. In this case it is in stable equilibrium; for when displaced slightly, it will tend to return to its original position.

An important consequence of the theorems of this article is the Principle of Least Work, explained in Art. 131, Chapter VI.

PROBLEM

211. The annular space in a hollow ring with closed ends (Fig. 116) is partly filled with mercury. Water is then poured in on the right side to a certain height while the ring is held stationary. If the ring rests on a smooth, level table, will motion occur when it is released, and if so, why will it not be perpetual?

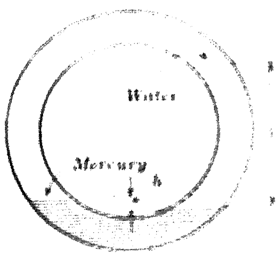


FIG. 116.

78. Structures: External Forces. — The external forces acting upon any stationary structure must be in equilibrium. Hence they may be found, in general, by applying the conditions of equilibrium given in Art. 74.

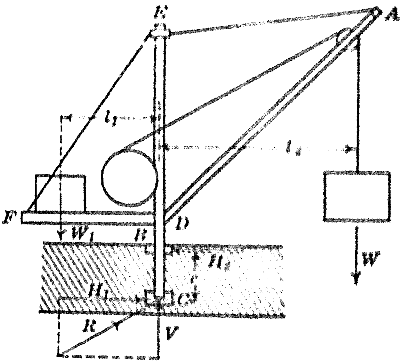


FIG. 117

The conditions of equilibrium may be applied either analytically or graphically. The former method has the advantage of being available under all circumstances, whereas the latter method requires the accurate use of instruments, and is therefore confined chiefly to office work. Both methods are illustrated in what follows.

I. *Analytical Method.* Consider first the analytical determination of the external forces acting on a simple structure, such as the loaded jib crane, shown in Fig. 117. This consists of a

CASE I. If $\left(\frac{\partial^2 P}{\partial s^2}\right)_1$ is positive, then $P - P_1$ must remain positive throughout the motion; that is, we must have $P > P_1$, because the other factor $\frac{s^2}{2}$ is essentially positive. Since P is decreasing, s must therefore also decrease. Hence when displaced from its position of equilibrium, the system moves back toward this position. The equilibrium in this case is said to be **stable.** Hence *for stable equilibrium the potential energy is a minimum.*

CASE II. If $\left(\frac{\partial^2 P}{\partial s^2}\right)_1$ is negative, then by the same reasoning as above $P - P_1$ must remain negative throughout the motion; that is, $P < P_1$. Therefore since P is decreasing, s must increase, and consequently when the system is displaced from its position of equilibrium and then released, it moves away from this position. The equilibrium in this case is called **unstable.** Hence *for unstable equilibrium the potential energy is a maximum.*

CASE III. If $\left(\frac{\partial^2 P}{\partial s^2}\right)_1 = 0$, then $P = P_1$; that is to say, the potential energy is not changed by the displacement, and consequently there is no tendency to move in either direction. The equilibrium in this case is called **neutral.** Hence *for neutral equilibrium the potential energy is constant.*

As a simple example of the different conditions of equilibrium, consider a heavy sphere resting on a level table. If the center of gravity of the sphere coincides with its center of figure, it may be placed in any position on the table and will show no tendency to move. It is then in neutral equilibrium. Its potential energy in this case is its weight multiplied by the distance of its center of gravity above a fixed horizontal plane of reference, and since neither factor is changed by the motion, its potential energy remains constant.

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An important consequence of the theorems of this article is the Principle of Least Work, explained in Art. 131, Chapter VI.

PROBLEM

211. The annular space in a hollow ring with closed ends (Fig. 116) is partly filled with mercury. Water is then poured in on the right side to a certain height while the ring is held stationary. If the ring rests on a smooth, level table, will motion occur when it is released, and if so, why will it not be perpetual?

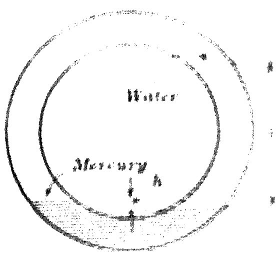


Fig. 116

78. Structures: External Forces. — The external forces acting upon any stationary structure must be in equilibrium. Hence they may be found, in general, by applying the conditions of equilibrium given in Art. 74.

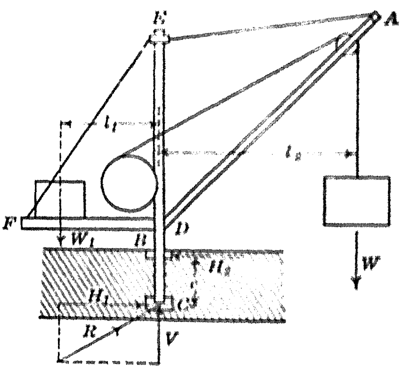


Fig. 117

The conditions of equilibrium may be applied either analytically or graphically. The former method has the advantage of being available under all circumstances, whereas the latter method requires the accurate use of instruments, and is therefore confined chiefly to office work. Both methods are illustrated in what follows.

I. *Analytical Method.* Consider first the analytical determination of the external forces acting on a simple structure, such as the loaded jib crane, shown in Fig. 117. This consists of a

vertical mast ED, supported by a collar B and footstep C, and carrying a jib AD, supported by the guy AEF. The external forces acting on the crane are the load W, the counterweight W_1, consisting of hoisting engine and machinery, and the reactions at B and C. The reaction of the collar B can have no vertical component, as the collar is made a loose fit so that the crane may be free to swivel. For convenience, the reaction of the footstep C may be replaced by its horizontal and vertical components H and V.

Applying the conditions of equilibrium to the structure as a whole, we have therefore

$$\sum \text{Vertical forces} = 0, \quad W + W_1 + \text{weight of crane} - V = 0,$$

$$\sum \text{Horizontal forces} = 0, \quad H_1 + H_2 = 0,$$

$$\sum \text{Moments} = 0 \text{ (taken about } B\text{)}, \quad Wl_2 - W_1 l_1 + H_1 e = 0.$$

From the first condition the vertical reaction of the footstep is found to be equal to the entire weight of the structure and its loads. In applying the last condition moments are taken about B since the unknown H_2 is thus eliminated, leaving the resulting moment equation with only one unknown H_1. The other unknown H_2 is then found from the second condition, $H_2 = -H_1$.

The moment of the counterweight $W_1 l_1$ should, when possible, be made equal to $\frac{Wl_2}{2}$, where W is the maximum load the crane is designed to lift. The mast will then never be subjected to a bending moment of more than one half that due to the lifted load; that is to say, the horizontal reactions H_1 and H_2 will never be more than one half the value they would have if the crane was not counterweighted.

II. *Graphical Method.* To illustrate this method, consider the Pratt truss shown in Fig. 118. Assume the loads in this case to be the weight of the truss W, a uniform load of amount W_1, assumed for present purposes to be concentrated at its center of gravity, and two concentrated loads P_1, P_2. Since the only other external forces acting on the truss are the reactions R_1, R_2, they must hold the loads in equilibrium, and hence the force

polygon must close. The force polygon, however, consists in the present case simply of a straight line 1 2 3 4 5, and therefore does not suffice to determine the values of R_1 and R_2. For this purpose an equilibrium polygon must be drawn. Thus choose any pole O on the force diagram, and draw the rays $O1$, $O2$, $O3$, etc., and then construct the corresponding equilibrium polygon by starting from any point a in R_1 and drawing ab parallel to $O1$; from b drawing bc parallel to $O2$, etc. Having found the closing side af of the equilibrium polygon, draw through O the ray $O6$ parallel to af, thereby determining R_1 as 56 and R_2 as 61.

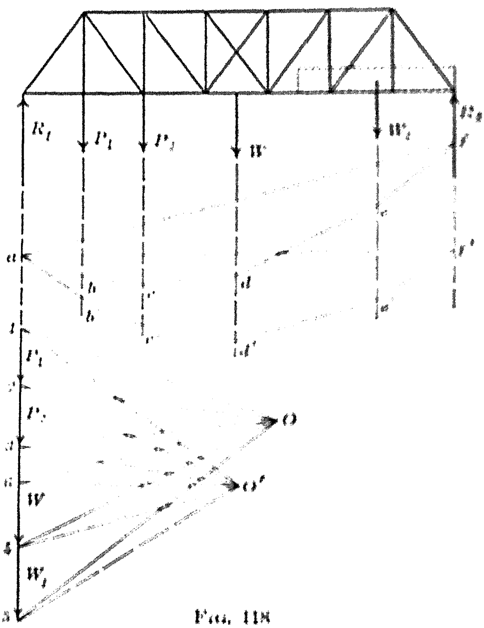


FIG. 118

If, for any reason, it is desired to draw the equilibrium polygon through two fixed points, say a and f' in the figure, the reactions (*i.e.* the point 6) are first determined as above. Then a line is drawn through 6 parallel to af', and a pole O' chosen somewhere on this line. The closing side of the equilibrium polygon will then necessarily be parallel to $O'6$ (or af'), and hence if the polygon starts at a, it must end at f'.

PROBLEMS

212. A ladder 50 ft. long and weighing 75 lb. rests with its upper end against a smooth vertical wall and its lower end on rough horizontal ground. Find the reactions of the supports when the ladder is inclined 20° to the vertical.

213. A circular, three-legged table, 4 ft. in diameter, weighs 50 lb. and carries a load of 100 lb. 10 inches from the center. Find the pressure between each foot and the floor. Find also the smallest load which when hung from the edge of the table will just cause it to tip over.

214. An engine is part way across a bridge, the weights and distances being as shown in Fig. 119. Find the reactions of the abutments.

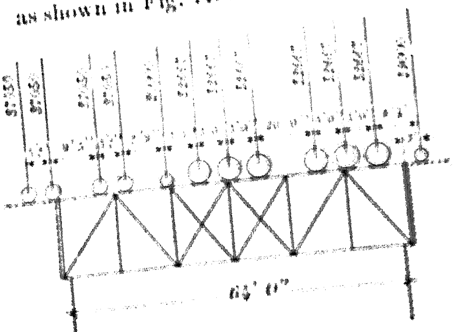


Fig. 119

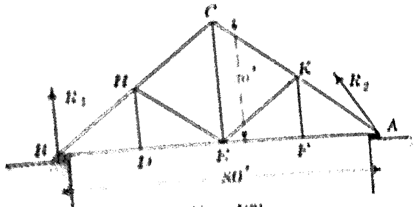


Fig. 120

215. The roof truss shown in Fig. 120 is anchored at one end A, and rests on rollers at the other end B. The span is $l = 80$ ft., rise $h = 30$ ft., distance between trusses $b = 18$ ft. The weight of the truss is given approximately by the formula $W = \frac{3}{4} bl^2$; the wind load, assumed to be from the left, is taken as 45 lb./ft.² of roof surface, and the snow load is 30 lb./ft.² of horizontal projection. Calculate analytically the reactions of the supports due to all loads acting on the truss.

216. Solve the preceding problem graphically.

217. Three smooth cylindrical water mains, each weighing 500 lb., are placed in a wagon box, two of them just filling the box from side to side and the third being placed on top of these two. Find the pressure between the pipes, and also against the bottom and sides of the wagon.

79. Structures: Joint Reactions. — Since all parts of a structure at rest are in equilibrium, the conditions of equilibrium may evidently be applied to the forces acting upon any portion of the structure. This portion may be a single joint, a single member or part of a member, or it may include several joints and members. The forces acting upon the part considered may be part external forces and part internal forces, or stresses, or they may be wholly stresses.

As in finding external reactions, the conditions of equilibrium may be applied either analytically or graphically.

Analytical Method. To illustrate this method, as applied to the joints of a structure, let it be required to find the stresses in the members of the shear legs, shown in Fig. 121.

Starting with the joint A, the forces acting at this point are

the weight W, the tension P in the guy AC, and the reaction of the legs of the A frame. To simplify the solution the latter may be assumed for the present equivalent to a single force R acting

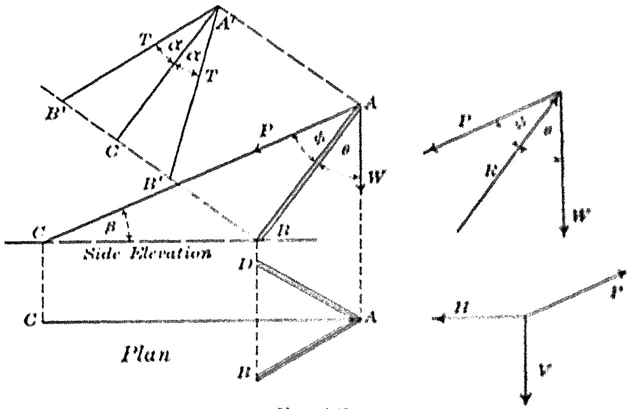


FIG. 121

along the center line $A'C'$ between the legs of the A frame. The conditions of equilibrium applied to this joint are then

$$\sum \text{Vertical forces} = 0,\ W + P\cos(\theta + \phi) - R\cos\theta = 0;$$

$$\sum \text{Horizontal forces} = 0,\ P\sin(\theta + \phi) - R\sin\theta = 0,$$

giving two simultaneous equations for R and P.

Since R is by assumption equivalent to the combined action of the shear legs, the thrust T in each may be found by resolving forces along R. Thus $T\cos\alpha = \frac{1}{2}R$, which determines T, since R has already been found.

Similarly the force at the bottom of the shear legs tending to make them spread is $T\sin\alpha$.

At the point C the forces acting are the upward pull V on the anchorage, the horizontal pull H on it, and the tension P in the guy. Hence applying the conditions of equilibrium

$$H = P\cos\beta,\ V = P\sin\beta.$$

Graphical Method. To illustrate the graphical calculation of stresses from joint reactions, consider the roof truss shown in Fig. 122.

Since the loading in this case is symmetrical, the reactions of the supports will each be equal to half the weight on the truss.

The most convenient notation is to letter the spaces between the various lines of the diagram. Each member of the truss and each external force will then be designated by the adjoining letters on opposite sides of it, as the member *AH*, the load *BC*, etc.

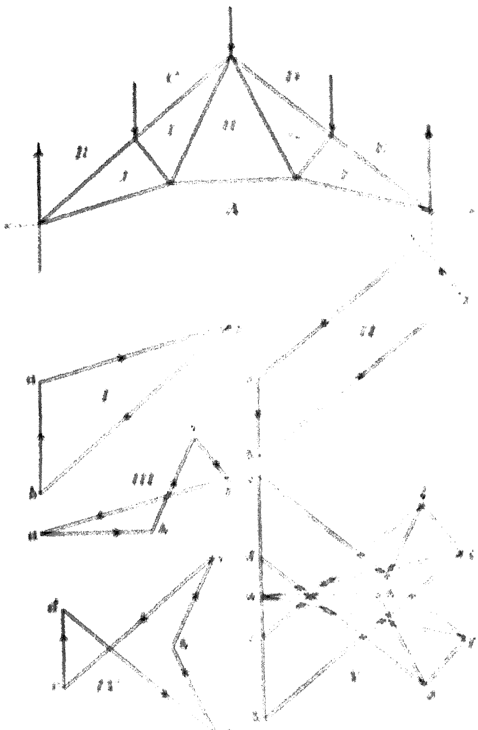


FIG. 122

Starting with the left support, we have three forces meeting at a point. The magnitude of one, namely, R_1, or *AB*, is known, and the directions of all three are known. Hence the other two can be determined by means of a triangle of forces. Thus if *ab* is laid off to scale to represent R_1, and *aj*, *bj* are drawn from *a* and *b* parallel to *AJ* and *BJ*, they will represent the stresses in these members to the same scale as that to which R_1 was laid off (Fig. 122 (*I*)).

Proceeding to the next joint, *BJIC*, we have four forces meeting at a point, one of which, *BJ*, has just been determined, and another, *BC*, is known. Hence the other two are found by drawing a force polygon, *bjic*, giving the stresses in *CI* and *IJ* (Fig. 122 (*II*)).

Similarly, passing to the next joint, *AJIH*, the stresses in *AJ* and *JI* having been found, those in *IH* and *AH* may be determined from the force polygon *ajih* (Fig. 122 (*III*)), and finally for the joint *HCD* the remaining stresses are determined from the force polygon *ghicd* (Fig. 122 (*IV*)).

Since each force polygon contains one side of each of the others, by placing these sides together they may all be combined into one figure, as shown in Fig. 122 (*V*). In the present case separate diagrams were drawn for each joint to illustrate the method. In practice, however, but one diagram, the combined one, is drawn, as it affords a saving in time and space and produces a neater and more compact appearance. Such a figure is called a **Maxwell diagram.**

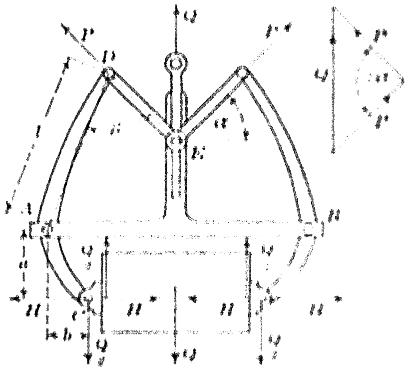


Fig. 123

PROBLEMS

218. In the crab hook shown in Fig. 123, assume that the load $Q = 300$ lb., the coefficient of friction $\mu = .5$, and determine the kind and amount of strain in the members ED, AD, and AB for $\alpha = 30°$, $\beta = 90°$, $a = 6$ in., $b = 3$ in., and $l = 18$ in. Show also that in order to hold the weight without slipping, the condition which must be satisfied is

$$\mu\left(\frac{l}{2a\sin\alpha} - \frac{b}{2a}\right) > \frac{1}{2}.$$

219. A steam cylinder is 20 in. in diameter, the steam pressure is 150 lb./in.², the crank is 18 in. long and the connecting rod is 5 cranks long. Find the stress in the connecting rod, pressure on cross-head guides, and tangential pressure on crank pin when the crank makes an angle of 45° with the horizontal on the "in end" of the stroke. Find also the maximum tangential pressure on the crank pin.

220. Find the cutting force exerted by the blades of the orange-peel bucket shown in Fig. 124 if the bucket weighs 4200 lb., its diameter when closed is 5 ft. 8 in., and its capacity is 1 cu. yd., weighing 3000 lb.

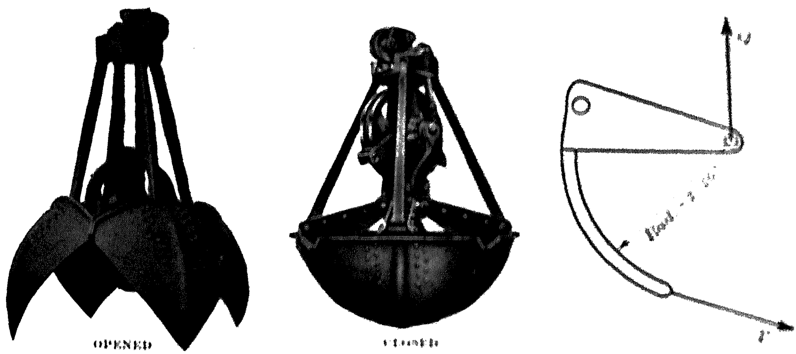


Fig. 124

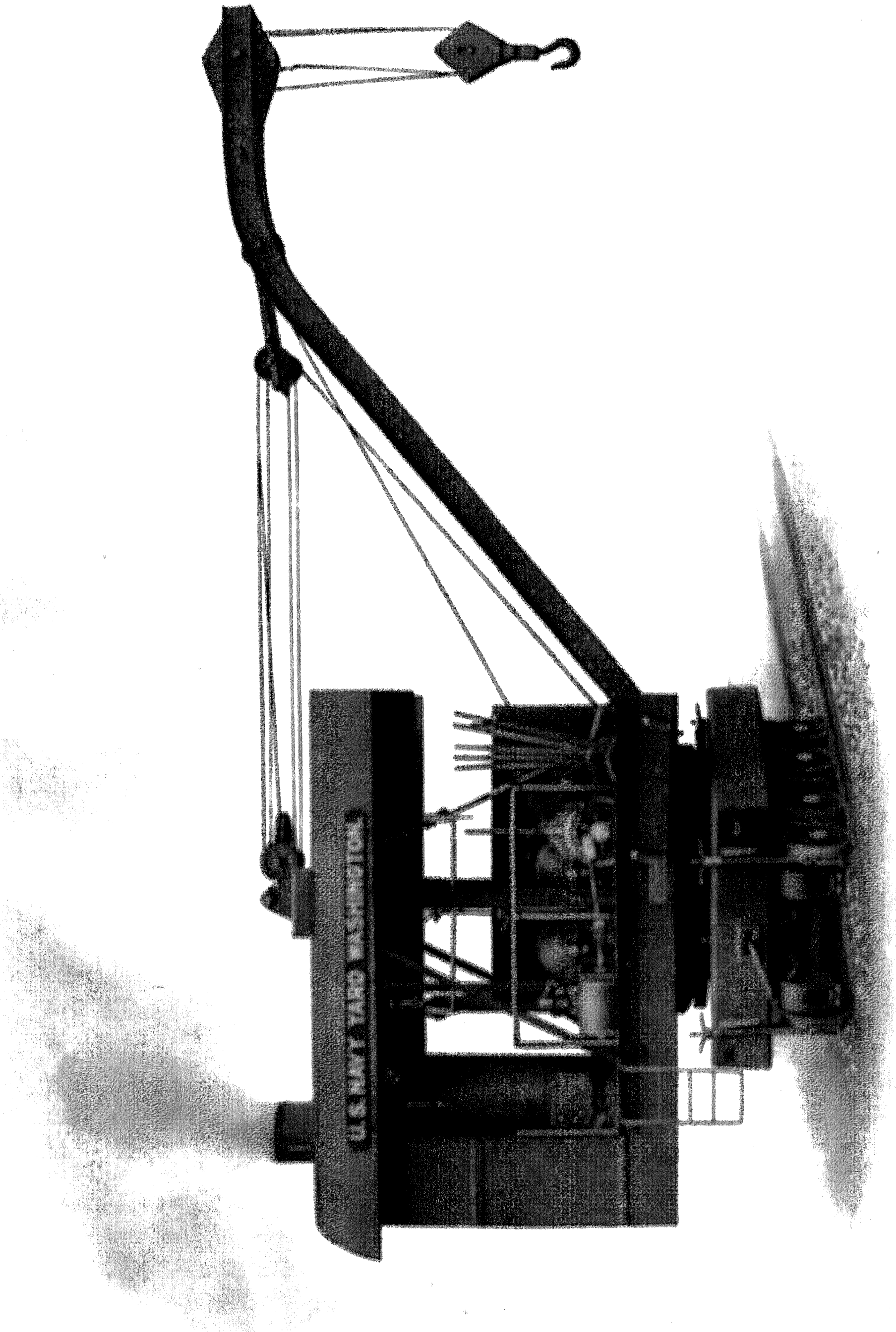


FIG. 125

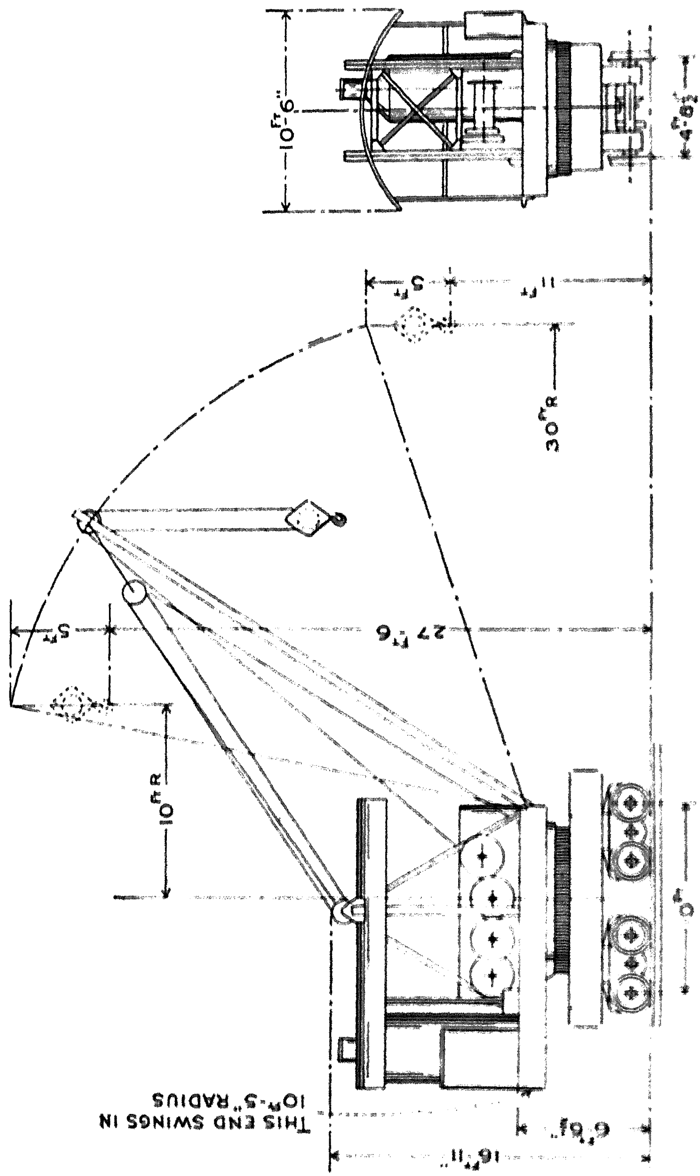

10FT-6"
4FT-8½"
5FT
11FT
30FT R
5FT
27FT 6
10FT R
THIS END SWINGS IN
10FT-5" RADIUS
6FT 6½"
16FT 11"

FIG. 135

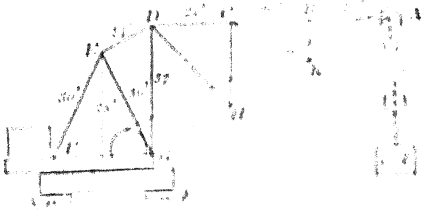


FIG. 127

221. Calculate the stresses in all the members of the dockyard crane shown in Fig. 127, when carrying a load of 10 T.

222. In the locomotive crane shown in Fig. 125, the dimensions are as shown in Fig. 126. Calculate the stresses in the boom and hoisting cables, and also in the mast, boom, and back stays, assuming the load to be 10 T., the boom inclined at 15° to the horizontal, the boom tackle horizontal, and the back stays inclined at 60° to the horizontal.

223. In the saw-tooth type of roof truss shown in Fig. 128, obtain graphically the stresses in all the members, the dimensions being as follows: Span = 25 ft., distance apart of trusses = 15 ft., and pitch of roof = $\frac{1}{4}$, making the inclination of the longer leg to the horizontal = 21° 48′.

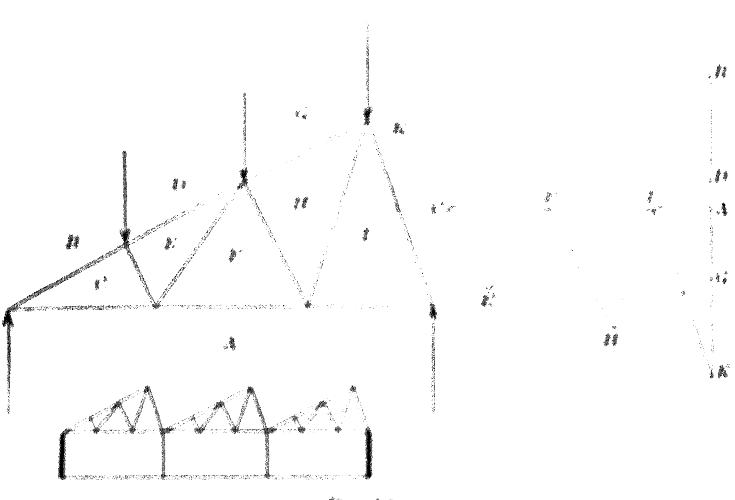


FIG. 128

As the span is short and the roof comparatively flat, it is sufficiently accurate to assume that the combined action of wind and snow is equivalent to a uniform vertical load, which in the present case may be assumed as 25 lb./ft.^{2} of roof. The weight of this type of truss will be taken as 1.5 lb./ft.^{2} of roof, and the weight of roof covering as 7 lb./ft.^{2} of roof. As the top chord panel length is 8 ft., each panel load will be $8 \times 15 \times 33.5 = 4020$ lb.

224. Analyze graphically for both dead and snow loads the French type of roof truss shown in Fig. 129, the span being $l = 100$ ft., rise $h = 30$ ft., $d = 5$ ft., and distance apart of trusses $b = 20$ ft.

The weight of truss in pounds for this type is given by the formula $W =$ [illegible] bl^2, where b and l are expressed in feet. The weight of the roof covering may be assumed as 15 lb./ft.^{2} of roof surface, and the snow load as 20 lb./ft.^{2} of horizontal projection.

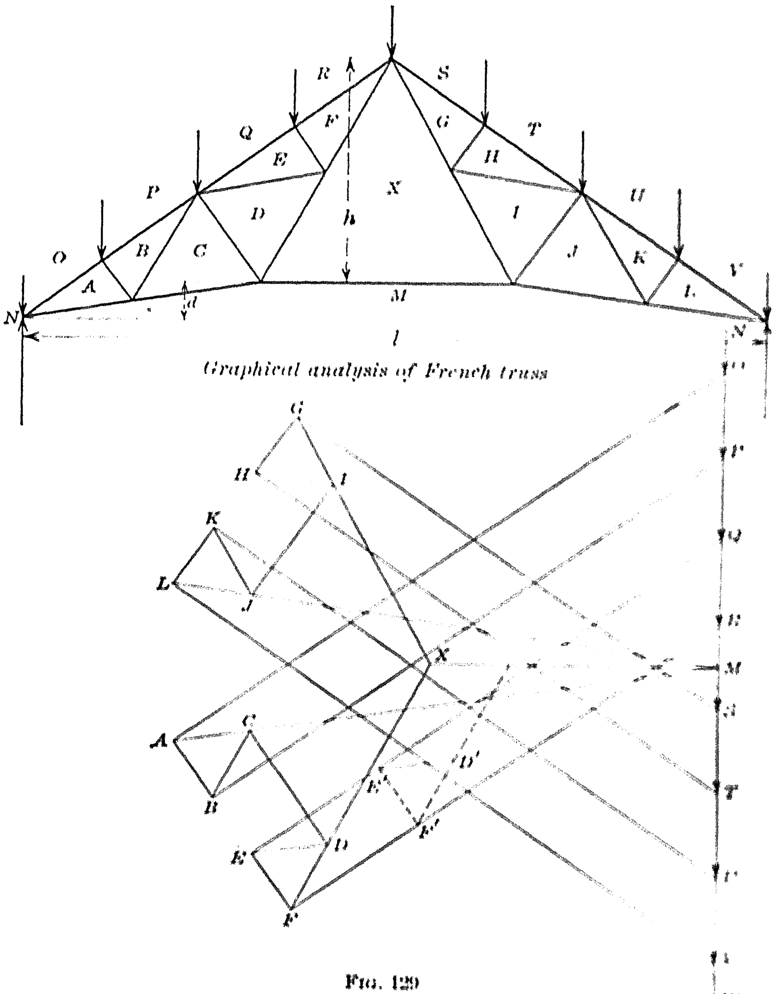


FIG. 129

NOTE. — First calculate the dead load carried at each joint, due to weight of truss and roof covering, and draw the diagram for this system of loads. The diagram for snow load will be a similar figure, and the stresses in the two cases will be proportional to the corresponding loads. Hence the snow load stresses may be obtained by multiplying each dead-load stress by a constant factor equal to the ratio of the loads.

In drawing the diagram start at one abutment, say the left, and take the joints in order, thus determining the stresses in OA, AM, AB, PB, BC, and CM. At the middle of the rafter, where the load PQ is applied, there will be three unknowns, and since these cannot all be determined simultaneously, one of the three must be obtained by some other means before the construction can proceed. For

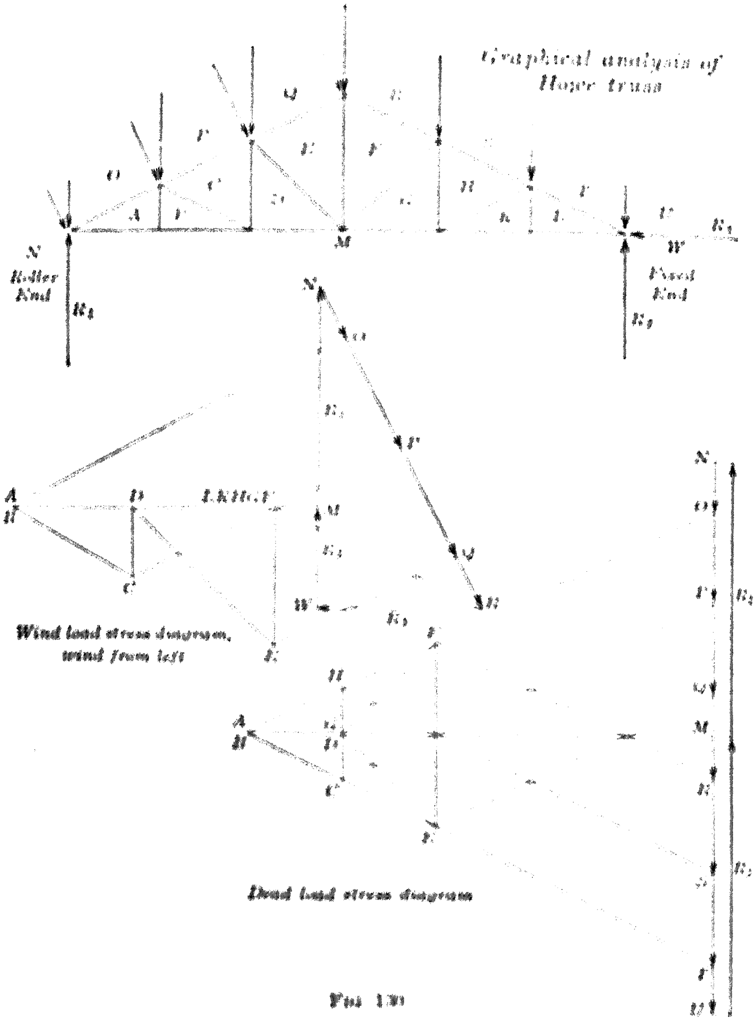


FIG. 130

this purpose, proceed to the load QR and determine the stress in EF by the auxiliary construction shown by the dotted lines in the diagram. Then determine the stress in ED by the same auxiliary construction. Having found the stress in ED, we may then go back to the load PQ and complete the diagram.

225. Analyze graphically for both dead and wind loads the Howe roof truss shown in Fig. 130 for a span of 50 ft. and quarter pitch, *i.e.* with a rise of $\frac{50}{4} = 12.5$ ft. The trusses are spaced 16 ft. apart; the weight of each truss may be taken as 2.5 lb./ft.^{2} of horizontal area; the roof covering as 10 lb./ft.^{2} of roof, and the snow load as 20 lb./ft.^{2} of roof. The wind load, based on a pressure of 30 lb./ft.^{2} of vertical projection, gives for a roof of one quarter pitch an equivalent load of 22.4 lb./ft.^{2} of roof surface.

Assume the left end of the truss to be on rollers and the right end fixed. The total horizontal thrust due to wind load is then carried by the right abutment. In drawing the wind-load diagram first calculate the reactions R_1 and R_2 by the method of moments. Having thus determined the point M, the remainder of the diagram is easily drawn.

80. Structures; Method of Sections. — If a section is passed through a structure, cutting not more than two members whose stresses are unknown, the single condition that the force polygon, drawn for the forces acting upon the portion of the structure on one side of the section, must close, will enable the stresses in these members to be found. Commencing at one end of a structure and passing a section cutting but two members, the stresses in these can thus be determined. Then passing a section cutting three members, one of which has already been treated, the stresses in the other two can be found, etc. Thus by means of successive sections, all of the stresses can be determined by simple force polygons.

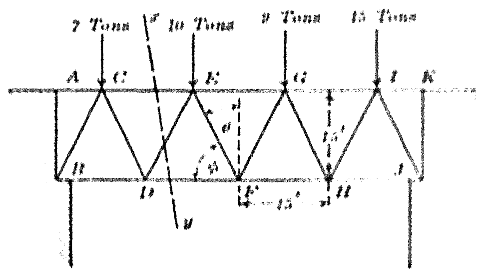


FIG. 131

Analytical Method. To illustrate the analytical application of this method, consider a Warren truss used as a deck bridge, as shown in Fig. 131. Let the depth of truss and panel length be each 15 ft., and the loads carried at the joints of the upper chord be 7, 10, 9, and 15 T., respectively. The reactions at B and J are found by taking moments around J and B to be $17\frac{5}{8}$ T. and $23\frac{3}{8}$ T. respectively.

Since this form of truss has parallel chords and a single web system, it is not necessary to begin at any particular point, but a section may be taken anywhere, provided it cuts both chords and

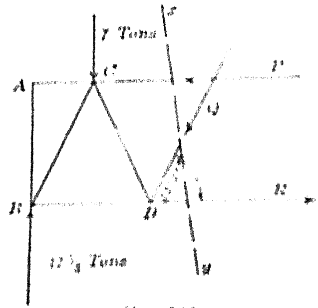


FIG. 132

a single web member. Taking any section xy, and considering only the portion of the structure on one side of the section, the external forces acting on this portion will be in equilibrium with the stresses P, Q, R in the members cut (Fig. 132). Since Q is the only stress having a vertical component, it must equilibriate the external forces at B and C. That is to say, from the condition of equilibrium

$$\sum \text{Vertical forces} = 0, \text{ we have } \quad Q \sin 63^\circ 26' = 17\tfrac{5}{8} - 7,$$

whence $Q = 11.854$ T., and is compressive.

To find P take moments about D. Then since Q and R both pass through D, their moments about this point are zero; therefore

$$P \times 15 = 17\tfrac{5}{8} \times 15 - 7 \times 7.5,$$

whence $P = 14\tfrac{1}{8}$ T. By observing the signs of the moments of the external forces at B and C about D, P is found to act in the direction shown by the arrow; that is, in compression.

Similarly to find the stress R in DF take the section xy just to the left of E, then take moments about E. Since P and Q pass through E, their moments about this point are zero, and hence

$$R \times 15 = 17\tfrac{5}{8} \times 22.5 - 7 \times 15,$$

whence $R = 19.44$ T.

Since the loads are vertical, R might also have been found from P and Q by the condition

$$\sum \text{Horizontal forces} = 0, \text{ i.e. } P + Q \cos 63^\circ 26' = R,$$

whence $R = 19.427$ T.

Graphical Method. Before proceeding with the explanation of the graphical method, it will be necessary to show how the moment of any number of forces with respect to a given point may be obtained from the equilibrium polygon.

Let P_1, P_2, P_3, P_4, denote any set of forces, and B the given point about which their moment is required (Fig. 133). First

draw the force polygon for these forces, choose any pole O and construct the corresponding equilibrium polygon *abcde*. Now in the force diagram, drop a perpendicular Oh from the pole O on the resultant R. This is called the pole distance of R, and will be denoted by H. Also, in the equilibrium diagram draw through the given point B a line parallel to R, making the intercept xy on the equilibrium polygon. Then the triangle OAE in the force diagram is similar to the triangle xey in the equilibrium diagram, and hence

$$r : xy = H : AE,$$

or

$$Rr = H \times xy.$$

But Rr is the moment of the resultant R about B and is equal to the sum of the moments of all the given forces about this point. The following moment theorem may therefore be stated:

The moment of any system of forces about a given point is equal to the pole distance of their resultant multiplied by the intercept made by the equilibrium polygon on a line drawn through the given point parallel to the resultant.

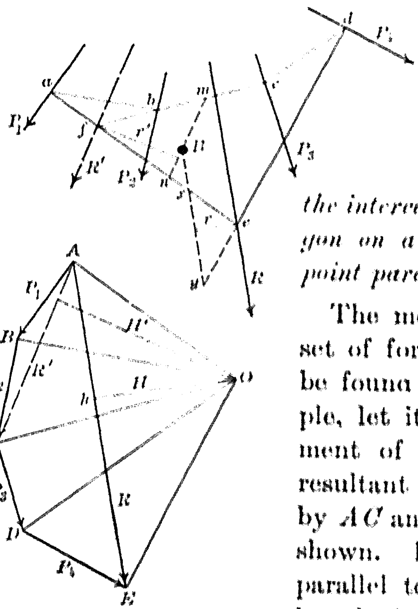


FIG. 133

The moment of a part of the given set of forces about any point may also be found by this theorem. For example, let it be required to find the moment of P_1 and P_2 about B. The resultant of P_1, P_2 is given in amount by AC and acts through the point f, as shown. Hence draw through B a line parallel to this partial resultant, making the intercept mn on the equilibrium polygon. Then since the triangles fmn and OAC are similar, we have

$$r' : mn = H' : R',$$

or

$$R'r' = H' \times mn,$$

which is the expression required by the theorem.

For a system of parallel forces the pole distance H is constant, and hence the equilibrium polygon is similar to the moment diagram for the forces on either side of any given point. Therefore the moment of all the forces on one side of a given point taken with respect to this point is equal to the constant pole distance H multiplied by the intercept made by the equilibrium polygon on a vertical through the point in question.

To apply this method to the roof truss shown in Fig. 134, for example, draw the force polygon and the corresponding equilibrium polygon, as shown in the figure. Now take any section of the truss, such as xy in the figure, and take moments of the stresses in the members cut about one of the joints, say B. Then the condition of equilibrium

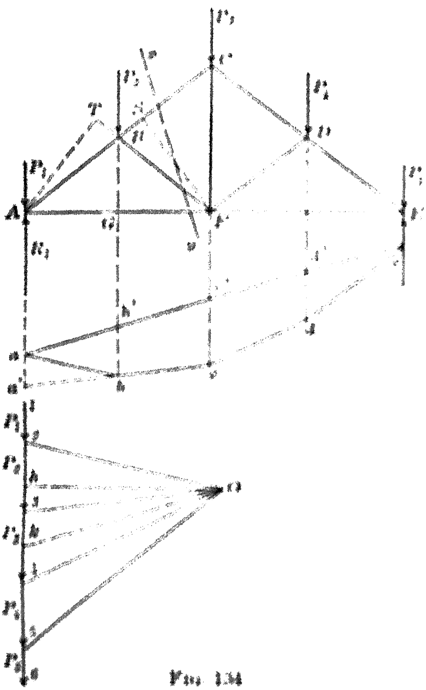


FIG. 134

$$\sum \text{Moments about } B = 0$$

may be written

$$\text{Moment of stress in } AF + \sum \text{Moments of } P_1, P_2, R_1 = 0.$$

But by the above theorem

$$\sum \text{Moments of } P_1, P_2, R_1 \text{ about } B = bb' \times Oh.$$

Hence

$$\text{Stress in } AF = \frac{bb' \times Oh}{BG}.$$

Similarly by taking moments about A the stress in BF is found to be

$$\text{Stress in } BF = \frac{aa' \times Oh}{AT},$$

and the stress in BC, with center of moments at F, is

$$\text{Stress in } BC = \frac{cc' \times Oh}{FS}.$$

By observing the signs of the moments, the stresses in AB, BC, and BF are found to be compressive, and in AF tensile.

In the present case, from symmetry, the stresses in the remaining members of the truss are the same as in those already found. For unsymmetrical loading it would be necessary to apply the above method to each individual member.

PROBLEMS

226. Determine analytically the stresses in the members CD, DE, and EF of the curved chord Pratt truss shown in Fig. 135, assuming the load at each panel point to be 50,000 lb.

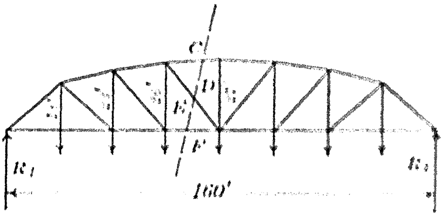


FIG. 135

227. Calculate analytically the stresses in the members of the jib crane shown in Fig. 136 when lifting a load of 28 T., the dimensions being as given in the figure.

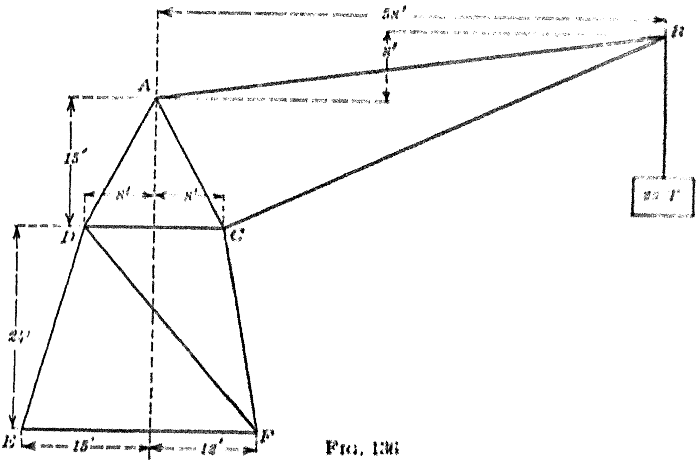


FIG. 136

228. In the saw-tooth type of roof truss shown in Fig. 128, determine analytically the stress in FH.

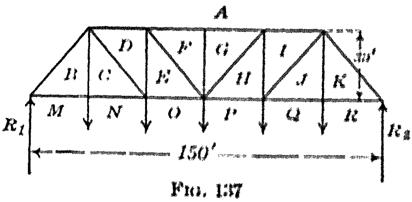


FIG. 137

229. In the Pratt truss shown in Fig. 137, the dimensions and loads are as follows: Span = 150 ft., height = 30 ft., number of panels = 6. The dead load per linear foot in pounds for single track bridge of this type is given by the formula $w =$ [illegible],

where l denotes the span in feet; the weight of single track may be taken as 400 lb. per linear foot, and live load as 3500 lb. per linear foot. Calculate analytically the stresses in all the members.

NOTE. Each truss carries one half the total load. In the present case, therefore, the total load per linear foot per truss is $\frac{1100 + 400 + 3500}{2} = 2500$ lb.

81. Flexible Cords. — Consider a flexible cord, fastened at the ends and supporting weights at various points of its length, as shown in Fig. 138. At any knot, say B, the weight W_2 and the tensions in the segments AB and BC constitute three forces in equilibrium, holding the point B stationary. Hence the cord itself forms an equilibrium polygon for the given system of loads. Consequently the pole O of the force diagram may be determined by laying off the loads W_1, W_2, etc., to scale along the line $abcdef$, as shown in Fig. 139, and drawing aO and fO from the initial and terminal points a and f parallel to MA and EN, respectively.

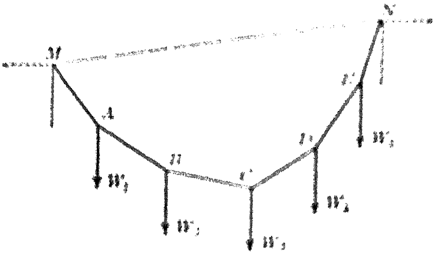


FIG. 138

Let the horizontal distance of O from the load line af be denoted by H. Then H represents the horizontal component of the tensions in each of the cords, which is therefore the same for all. Consequently if T_n and T_{n+1} denote the tensions in two successive segments of the cord, and α_n, α_{n+1} their inclinations to the horizontal (Fig. 139), then

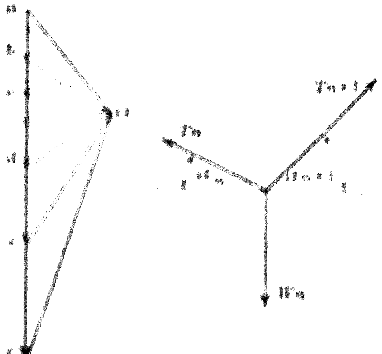


FIG. 139

$$H = T_n \cos \alpha_n = T_{n+1} \cos \alpha_{n+1}.$$

From the condition $\sum$ Vertical forces $= 0$ we also have

$$T_n \sin \alpha_n + T_{n+1} \sin \alpha_{n+1} = W_n.$$

Eliminating T_n and T_{n+1} between these two relations, the relation between the inclinations of successive segments is found to be

$$\tan \alpha_{n+1} = \frac{W_n}{H} - \tan \alpha_n.$$

PROBLEMS

230. The rule used by the makers of cableways for finding the stress in the cable is to calculate a factor $= \frac{\text{one half the span}}{\text{twice the sag}}$, and multiply the load, assumed to be at the middle, by this factor. Show how this formula is obtained, and how nearly it is correct.

231. It is usual to allow a sag in a cable equal to one twentieth of the span. What does the numerical factor in the preceding problem become in this case, and how does the tension in the cable compare with the load?

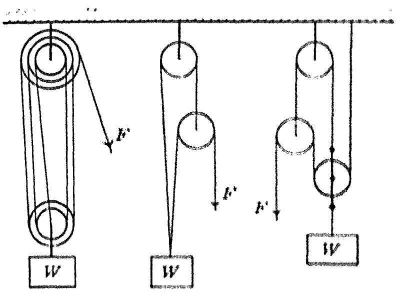


FIG. 140

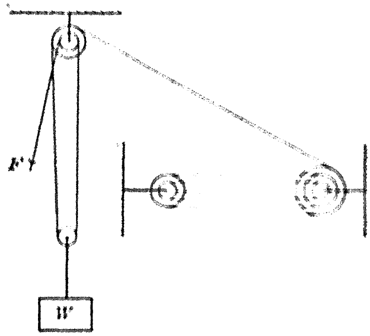


FIG. 141

232. Find the relation between F and W, and the total pull on the upper supports in the systems of pulleys shown in Fig. 140.

233. Find the relation between F and W, in the compound tackle shown in Fig. 141.

234. Find the stress in hoisting line and boom line in the quarry derrick shown in Fig. 143, assuming the effective length of mast and boom to be each 20 ft., load 15 T., and boom inclined at 45° to the horizontal.

235. In a Weston Differential Pulley two sheaves, of radii a and b, are fastened together, and by means of a continuous cord passing around both and also around a movable pulley, support a weight W. Find the relation between F and W, neglecting friction (Fig. 142).

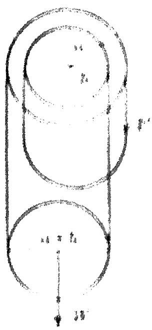


FIG. 142

236. In a Weston Differential Pulley the diameters of the sheaves in the upper block are 8 in. and 9 in. Find the theoretical advantage.

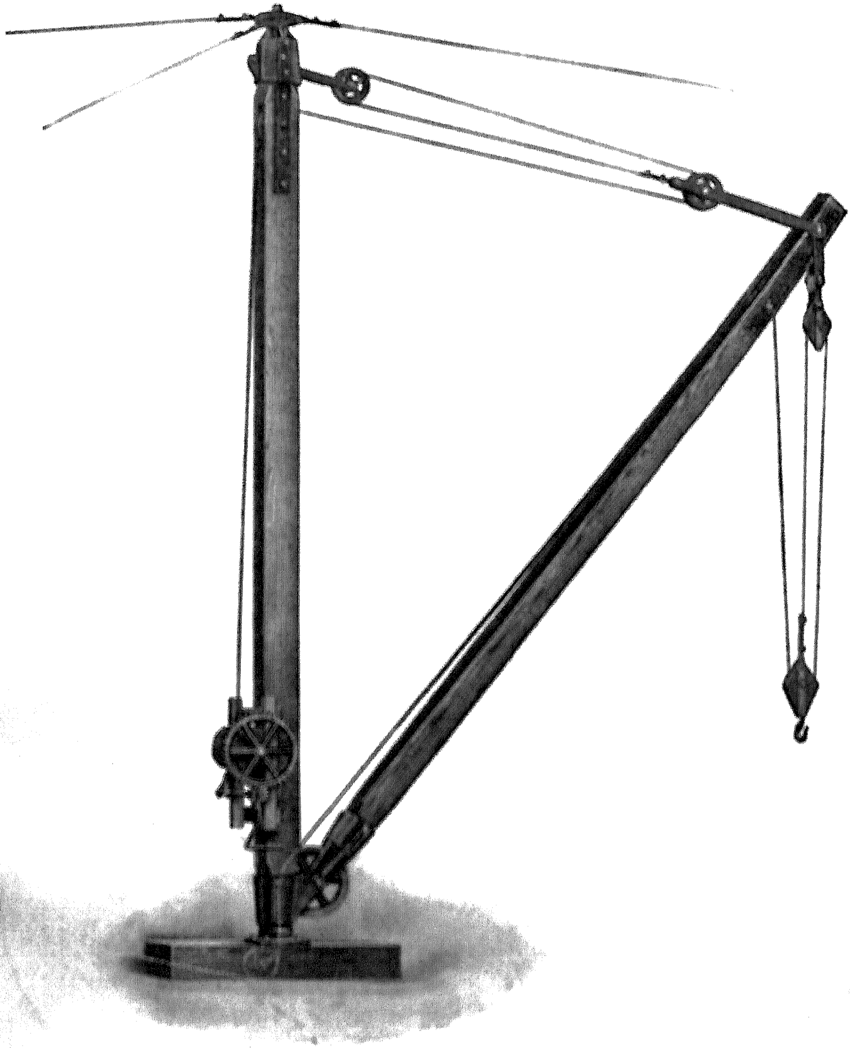


FIG. 143

237. Two equal weights of 50 lb. each are joined by a cord which passes over two pulleys in the same horizontal line, distant 12 ft. between centers. A weight of 5 lb. is attached to the string midway between the pulleys. Find the sag.

238. Find the mechanical advantage in the differential wheel and axle shown in Fig. 144 if the radius of the large drum is R, of the small drum r, and of the crank c.

NOTE. — The rope is wound in opposite directions around the two axles, so that it unwinds from one and winds up on the other at the same time.

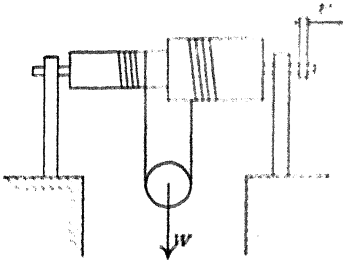


FIG. 144

82. Uniform Horizontal Load: Parabola. — When a flexible cord supports a load which is uniformly distributed over the horizontal projection of the cord, as, for example, the cable of a suspension bridge, which supports a load distributed uniformly per foot of roadway, the curve assumed by the cord is a parabola.

This is evident geometrically. For consider a portion of the cord OB, Fig. 145, O being the lowest point of the cord. Then the external forces acting on the cord are the tensions H and T at the ends of the part considered, and the weight of the load, acting at the center of OC. For three forces to be in equilibrium, however, they must meet in a point. Hence the tangent T to the cord at B passes through the middle of OC. However, it is a property of the parabola that the subtangent is bisected at the vertex, in which case $OD = \frac{1}{2} AB = DC$. Consequently the curve assumed by the cord is a parabola.

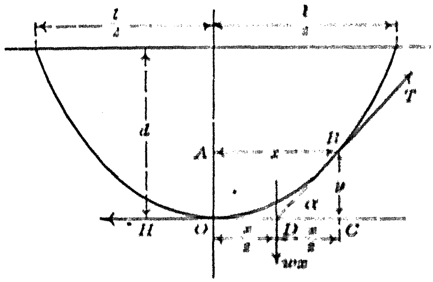


FIG. 145

This result may also be deduced by applying the conditions of equilibrium. Thus from

$$\sum \text{Horizontal forces} = 0, \quad H - T\cos\alpha = 0,$$

$$\sum \text{Vertical forces} = 0, \quad wx - T\sin\alpha = 0,$$

where w denotes the load per unit horizontal distance. Therefore, by division,

$$\tan\alpha = \frac{wx}{H},$$

and since from the figure we also have

$$\tan \alpha = \frac{2y}{x},$$

by equating these values of $\tan \alpha$,

$$y = \frac{wx^2}{2H},$$

which is the equation of a parabola referred to the vertex O as origin.

From the last equation it is evident that if the supports are on the same level, the sag d at the center is given by

$$d = \frac{wl^2}{8H},$$

where l denotes the length of the entire span.

The length of the cord for a given span and sag is found by applying the calculus formula for the length of any arc, namely,

$$s = \int \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx,$$

to the equation just found, $y = \frac{wx^2}{2H}$. In the present case $\frac{dy}{dx} = \frac{wx}{H}$ and consequently

$$s = 2\int_0^{\frac{l}{2}} \frac{w}{H}\sqrt{x^2 + \frac{H^2}{w^2}}\,dx,$$

which integrates into

$$s = \left[\frac{wx}{H}\sqrt{x^2 + \frac{H^2}{w^2}} + \frac{H}{w}\log_e\left(x + \sqrt{x^2 + \frac{H^2}{w^2}}\right)\right]_0^{\frac{l}{2}}$$

$$= \frac{wl}{2H}\sqrt{\frac{l^2}{4} + \frac{H^2}{w^2}} + \frac{H}{w}\log_e\left(\frac{l}{2} + \sqrt{\frac{l^2}{4} + \frac{H^2}{w^2}}\right),$$

or, since $H = \frac{wl^2}{8d}$, this expression finally becomes

$$s = 2d\sqrt{1 + \left(\frac{l}{4d}\right)^2} + \frac{l^2}{8d}\log_e\frac{l}{2}\left(1 + \sqrt{1 + \left(\frac{l}{4d}\right)^2}\right).$$

PROBLEMS

239. The International Railway Suspension Bridge built at Niagara Falls in 1854–55 was the first railway suspension bridge ever constructed. The data were as follows: Length of main span 800 ft., height of towers above floor 80 ft., number of cables 4, each composed of 3640 wires .148 in. in diameter. Combined strength of cables 12,000 T., permanent load on main cables 1000 T. Find the maximum tension in the cables due to this load.

240. A small suspension bridge of 80-ft. span carries a total uniform load of 35 T. The sag of the cables is 6 ft. Calculate the stress in the cables at the center and at the towers.

83. Heavy Cord: Catenary. — Consider a heavy flexible cord supporting only its own weight. Let A denote the lowest point of the cord, B any other point, s the length of AB, and w the weight of the cord per foot of length (Fig. 146). Then if H and T denote the tensions in the cord at A and B, respectively, the conditions of equilibrium give the equations

$$\left.\begin{aligned} H &= T\cos\phi, \\ ws &= T\sin\phi, \end{aligned}\right\} \qquad (27)$$

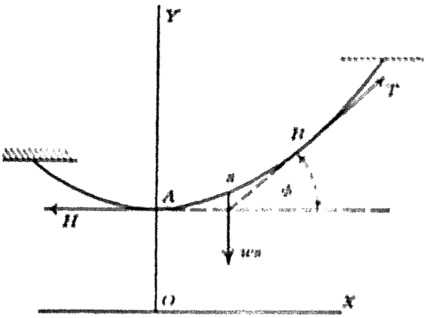


Fig. 146

whence

$$\tan\phi = \frac{ws}{H}.$$

Also since

$$\tan\phi = \frac{dy}{dx},$$

the differential equation of the curve is

$$\frac{dy}{dx} = \frac{ws}{H}. \qquad (28)$$

Let $y' = \frac{dy}{dx}$; then $y' = \frac{ws}{H}$, and therefore, by differentiating this expression,

$$dy' = \frac{w}{H}\,ds = \frac{w}{H}\sqrt{1 + y'^2}\,dx,$$

since $$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx.$$

Writing this equation in the form

$$\frac{dy'}{\sqrt{1 + y'^2}} = \frac{w}{H}\,dx,$$

it integrates into

$$\log_e(y' + \sqrt{1 + y'^2}) = \frac{wx}{H} + c_1.$$

Take for origin a point O at a distance equal to $\frac{H}{w}$ below the vertex of the curve. The constant c_1 may then be determined from the conditions $y' = 0$ $\left(i.e. \ \frac{dy}{dx} = 0\right)$, where $x = 0$. Hence $c_1 = 0$. Then writing the equation in the exponential form it becomes

$$y' + \sqrt{1 + y'^2} = e^{\frac{wx}{H}},$$

whence, solving for y',

$$y' = \frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{wx}{H}} - e^{-\frac{wx}{H}}\right). \tag{29}$$

Integrating this equation, the result is

$$y = \frac{H}{2w}\left(e^{\frac{wx}{H}} + e^{-\frac{wx}{H}}\right). \tag{30}$$

The curve represented by this equation is called the **catenary**.

For purposes of calculation it is more convenient to express Eq. (29) and (30) in terms of the hyperbolic functions. For this purpose let $\frac{H}{w} = c$. Then since

$$\tfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \sinh\frac{x}{c},$$

and

$$\tfrac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) = \cosh\frac{x}{c},$$

Eq. (29) and (30) become

$$\left.\begin{aligned} y' &= \sinh\frac{x}{c}, \\ y &= c\cosh\frac{x}{c}, \end{aligned}\right\} \tag{31}$$

$$c = \frac{H}{w}.$$

For the solution of practical problems several other relations are necessary. First, by equating the values of y' or $\frac{dy}{dx}$ given in (28) and (29), and solving the resulting expression for s,

$$s = \frac{H}{2w}\left(e^{\frac{wx}{H}} - e^{-\frac{wx}{H}}\right), \tag{32}$$

or

$$s = c\sinh\frac{x}{c}.$$

Squaring Eq. (30) and (32) and subtracting one from the other, the result is

$$y^2 = s^2 + \frac{H^2}{w^2}. \tag{33}$$

Adding Eq. (30) and (32),

$$y + s = \frac{H}{w} e^{\frac{wx}{H}},$$

whence

$$x = \frac{H}{w} \log_e \frac{w(y+s)}{H}. \tag{34}$$

Squaring Eq. (27) and adding, the result is $T^2 = H^2 + w^2s^2$, or since from (33) $H^2 + w^2s^2 = w^2y^2$,

$$T = wy. \tag{35}$$

If the sag is small as compared with the span, the catenary is approximately parabolic, and the simpler equations of the preceding article may therefore be used to give an approximate solution; namely,

$$d = \frac{wl^2}{8H}, \quad T \cos \alpha = H, \quad \tan \alpha = \frac{2y}{x}.$$

PROBLEMS

241. A rope weighing $\frac{1}{4}$ lb./ft. and 200 ft. long hangs from two supports not on the same level, the pull at these points being 60 lb. and 90 lb., respectively (Fig. 147). Find the sag and the span, or horizontal distance between supports.

Solution. (I) From (35), $T_1 = wy_1$ and $T_2 = wy_2$, from which find y_1 and y_2.

(II) From (33), $s_1^2 + c^2 = y_1^2$ and $s_2^2 + c^2 = y_2^2$. Also $s_1 + s_2 = 200$. From these three equations find s_1, s_2, and c.

(III) From (34),

$$x_2 = c \log_e \frac{y_2 + s_2}{c}$$

and $-x_1 = c \log \frac{y_1 + s_1}{c}$, from which find x_1 and x_2.

(IV) The results are then found from

$$\text{span} = x_1 + x_2,$$
$$\text{sag} = y_2 - c.$$

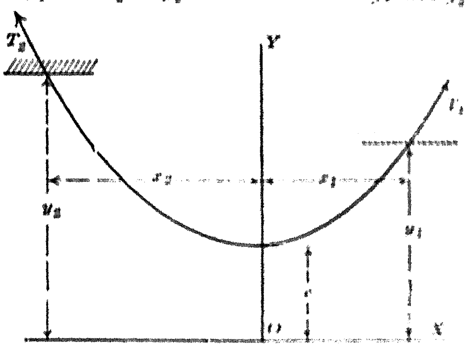


Fig. 147

242. Given the span, weight per foot, and tension at the ends, assuming that they are on the same level, find an expression for the sag.

Solution. Let P denote the end pull and $2a$ the span. Then $P = wy$ and $y^2 = s^2 + c^2$. Hence the sag

$$d = y - c = y - \sqrt{y^2 - s^2} = \frac{P}{w} - \sqrt{\frac{P^2}{w^2} - s^2}.$$

If the sag is small, the catenary may be assumed to be parabolic. In this case, since P is the resultant of the horizontal tension H and the load wa, we have as an approximate expression for the sag

$$d = \frac{wa^2}{2H} = \frac{wa^2}{2\sqrt{P^2 - w^2a^2}}$$

243. A rope of a rope drive weighs 1 lb./ft. Tension in driving side = 1500 lb., and in slack side = 500 lb. Distance between centers of pulleys = 300 ft. Find the sag for each half.

244. In the electric power transmission of the Bay Counties Power Co., across the Straits of Carquinez in California, four Roebling double galvanized steel strand cables are used, each $\frac{7}{8}$ in. in diameter and weighing 1.2 lb./ft. Total clear span 4427 ft., sag at center 190 ft. Find the maximum stress in the cable.

245. The cableway of the New River Coke Co. across the New River at Caperton, W. Va., consists of two main cables each 2 in. in diameter and weighing $6\frac{1}{4}$ lb./ft., with a clear span of 2077 ft. It operates by gravity from the mouth of a mine 500 ft. above the discharging terminal. Find the deflection due to the weight of the cable alone, and the total deflection when carrying a gross load of $3\frac{1}{2}$ T.

246. A hundred foot steel tape weighing .005 lb./ft. has its ends supported at the same level and is subjected to a pull of 10 lb. at each end. Find the error in measurement.

247. The Balanced Cable Crane shown in Fig. 148 consists of a cableway with counterweights suspended from inclined shear legs pivoted at the lower end. The weight of cable and car is thus automatically held in equilibrium by the counterweights and oscillating shears.

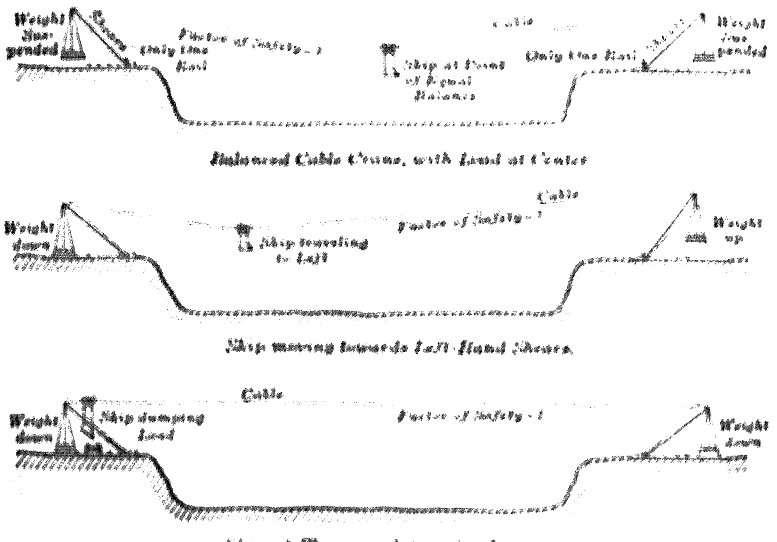


FIG. 148

Given the following data: Counterweights each 66,000 lb., inclination of shear legs 45° when cable is unloaded, horizontal projection of cableway in this position 525 ft., weight of cable 6.32 lb./ft. Find the tension and deflection for the cable and stresses in the shear legs when cable is unloaded, and also the tension and deflection when carrying a gross load of 10,000 lb. at the middle.

248. Solve the preceding problem for a cable weighing 3.71 lb./ft., span 1200 ft., and gross load of 8000 lb. at middle. Counterweight = 25 T.

249. Solve Prob. 247 for a cable weighing 3.71 lb./ft., span 900 ft., and gross load of 12,000 lb. at center. Counterweight 25 T.

CHAPTER IV

FRICTION AND LUBRICATION

84. Coefficient of Friction. One of the most important phenomena of mechanics is that known as friction. The common definition of friction is that it is the resistance to motion which is experienced when one body is caused to slide over the surface of another. By body in this case is meant any substance, whether solid, liquid, or gaseous. The chief property of friction is that it always tends to resist motion. In this respect it resembles inertia, since it always acts in opposition to external applied force. Since friction always acts in opposition to motion, it constantly absorbs work, thus dissipating mechanical energy in wear and heat losses. How to avoid or lessen such losses by proper design and lubrication is, then, one of the chief concerns of the mechanical engineer.

When two bodies touch, a force is, in general, transmitted between the surfaces in contact. If this force is resolved into components, one normal to the surfaces in contact and the other tangential, the latter is called the friction. As the force transmitted between the surfaces in contact changes, its tangential component also changes. The friction thus accommodates itself to the applied force, always entering in such amount as to prevent sliding, subject to the restriction that it cannot overstep a certain limiting value. In other words, if friction can prevent relative movement of the two bodies without overstepping its limiting value, the bodies will remain at rest. If, however, relative motion of the bodies actually occurs, friction takes the greatest value possible under the circumstances.

It is evident from what precedes that the friction increases with the normal pressure between the surfaces. The ratio of the limiting friction to the corresponding normal pressure is called **the coefficient of friction**. Thus, if the limiting friction is denoted

by F and the corresponding normal pressure by N, the coefficient of friction μ is defined by the ratio

$$\frac{F}{N} = \mu.$$

To illustrate, suppose that a block of weight W is placed on an inclined plane, and that the plane is then gradually tipped up until the block just begins to slide (Fig. 149). The weight W may be resolved into a normal component N and a tangential component F. The latter is equal and opposite to the friction, and when the block begins to slide, it becomes equal in amount to the limiting friction. Hence, the coefficient of friction is $\mu = \frac{F}{N}$, as above.

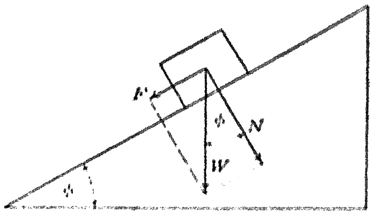


FIG. 149

From Fig. 149, $\frac{F}{N} = \tan \phi$, where ϕ is the inclination of the plane. Hence,

$$\mu = \tan \phi.$$

The angle ϕ is therefore called the **angle of friction** or **angle of repose.** The body will not slide if the inclination of the plane is less than the friction angle; if the inclination is just equal to the friction angle, it will be just on the point of moving; and if the inclination is greater than the friction angle, the body will be accelerated, the accelerating force in this case being equal to the difference between the tangential component of the weight and the limiting friction.

PROBLEMS

250. A weight of 400 lb. resting on a horizontal plane requires a horizontal force of 90 lb. to overcome friction. What is the value of the coefficient of friction?

251. A weight of 40 lb. is just supported by friction on an inclined plane of inclination 15° with the horizontal. How much is the friction?

252. A body resting on a horizontal plane requires a force equal to one third its own weight to overcome friction. If the plane is gradually tilted, at what angle will the body begin to slide?

253. On an inclined plane it is found that a body is just supported by a horizontal force equal to two thirds the weight of the body. Find the coefficient of friction.

254. Two unequal weights W_1 and W_2 are connected by a string which passes over a fixed pulley in the plane on which the bodies move, such that one body moves up the plane when the other slides down. If the coefficients of friction for the two bodies are μ_1 and μ_2 respectively, find the inclination of the plane when the system is in equilibrium.

85. Static Friction. The laws of friction have been experimentally determined at various times by a number of different physicists and engineers. Among the earlier experimenters the most notable were Coulomb and Morin. Their results applied only to dry surfaces under low speeds and pressures. Under these conditions the friction between two surfaces was found to be

Dependent on the nature of the material;
Independent of the areas in contact;
Directly proportional to the normal pressure;
Greater for rest than for motion.

Average values of the coefficient of friction are given in the following table. They should, however, be used with caution, as the actual value of the coefficient in any particular case may differ considerably from the values here given.

Material	Condition	Coefficient	
		Rest	Motion
Cast iron on cast iron or bronze	greasy	.16	.15
Wrought iron on wrought iron	dry		.44
	greasy	.14	
Wrought iron on cast iron or bronze	dry	.19	.18
Steel on steel	dry	.15	
Steel on ice			.02
Bronze on bronze	dry		.20
Bronze on wrought iron	greasy		.16
Bronze on cast iron	dry		.21
Oak on oak, fibers parallel to motion	dry	.62	.48
	soaped	.44	.16
Oak on oak, fibers perpendicular to motion	dry	.54	.34
	wet	.71	.25
Leather belts on oak	greasy	.47	
	dry		.27
Leather belts on cast iron	greasy	.28	

The distinction between static and kinetic friction was first pointed out by Coulomb (1736–1806), and the laws of static friction were also stated by him and confirmed by the experiments of Morin at Metz, in 1837–38. The low value of the coefficient of steel on ice, well known to every one who has used a pair of skates or a sled, is especially remarkable, and it is suggested by Föppl that a search be made among the crystals for some substance of similar properties.

PROBLEMS

255. A train of total weight 150 T. attains a speed of 40 mi./hr. from rest in one minute. If the coefficient of adhesion is .17, find the necessary weight on the drivers.

256. A brake shoe is pressed against a wheel with a force of 4 T. If the coefficient of friction is .25, find the h. p. absorbed by the brake when the car is traveling at 30 mi./hr.

257. A brake on a rotating pulley 4 ft. in diameter and revolving at a speed of 150 r. p. m. consists of a stationary band passing over it. If the tension in the tight side of the band is 150 lb. and in the slack side 30 lb., find the h. p. absorbed by the brake.

258. A locomotive has a total weight on the drivers of 44 T. and the coefficient of adhesion between wheels and rails is .15. What is the greatest draw-bar pull it can exert? Also how long will it take this engine to accelerate a train of 200 T. to 40 mi./hr. from rest if the total resistances amount to 20 lb./T.?

259. A ladder 20 ft. long and weighing 90 lb. rests at an angle of 30° with the horizontal against a rough vertical wall, the coefficient of friction being .6 at each end. How far can a man weighing 150 lb. mount the ladder before it will slip?

260. The inclination of a plane to the horizontal is 60° and the coefficient of friction is .5. How much longer will it take a body to slide down this plane than if it was perfectly smooth?

261. A drawer in a bureau is of length l and the coefficient of friction is μ. Show that the drawer can be pulled out by one handle if the distance of this handle from the center does not exceed $\frac{l}{2\mu}$.

86. Friction of Lubricated Surfaces. — From experiments made by Tower,* Goodman,† Dettmar,‡ Thurston, and others on lubri-

* *Proc. Inst. Mech. Eng.*, 1883, p. 632; 1884, p. 29; 1885, p. 58.
† *Eng. News*, Apr. 7 and 14, 1888.
‡ *Electrotechn. Zeitschr.*, 1899, p. 380.

cated bearings at various speeds, pressures, and temperatures, it was found that for well-lubricated bearings the laws of friction were almost exactly the reverse of those for static friction. Tower has shown that in a flooded bearing there is no metallic contact between the surfaces. The friction in this case is therefore independent of the material of which the bearing is made, and depends instead on the viscosity of the lubricant, and hence on the temperature. The friction of rest is also much greater than for motion, due to the fact that the lubricant is squeezed out when the bearing is allowed to stand for any length of time. Furthermore, dry surfaces gradually become polished, and hence the friction gradually decreases with the time, whereas with lubricated surfaces the lubricant gradually wears out, or is squeezed out, and hence the friction increases with the time. For convenience of comparison, the laws of friction for dry and lubricated surfaces are given in parallel columns in the following table:

LAWS OF FRICTION

DRY SURFACES	WELL-LUBRICATED SURFACES
The friction is:	The friction is:
1. Proportional to the normal pressure.	1. Independent of the pressure.
2. Independent of the speed for low pressures.	2. Varies directly as the speed for low pressures.*
3. Independent of the temperature.	3. Dependent on the temperature.
4. Dependent on the nature of the material.	4. Independent of the nature of the material.
5. Slightly greater for rest than for motion.	5. Much greater for rest than for motion.
6. A maximum at first and decreases with the time.	6. A minimum at first and increases with the time.

87. Friction Angle and Friction Cone. — As explained in Art. 84, the resultant of the normal pressure and the friction between two

* For high pressures the friction is very great at low speeds, becoming a minimum at a speed of about 100 ft./min. and afterward increasing approximately as the square root of the speed. In the latter case the work absorbed by friction increases as the 1.5 power of the speed.

surfaces forms a certain angle with the normal, which assumes its greatest value when the friction is a maximum. This maximum angle is called the **angle of friction**, and will be denoted by ϕ. The tangent of this angle is the coefficient of friction μ; that is,

$$\mu = \tan \phi.$$

All lines making the angle ϕ with the normal lie on the surface of a circular cone called the **friction cone**. By the use of the friction angle and friction cone, many problems in equilibrium are more easily solved than they could be otherwise, since the condition for equilibrium is simply that the resultant pressure at the surface of contact must lie within, or, at most, on the surface of, the friction cone.

The use of the friction cone is best explained by applying it to a few simple problems.

In Fig. 150 let AB represent a beam or ladder resting on rough supports at A and B, and let W denote its weight, or the resultant of all the weights acting on it. Draw the normals AC and BD to the surfaces in contact, and on either side of these normals lay off the friction angles ϕ_1 and ϕ_2. The four lines so drawn then form a quadrilateral, as shown by the shaded area in the figure. If, then, the resultant weight W cuts this quadrilateral, the ladder will be in equilibrium; for in this case, a point O may be chosen on the line of action of W and within the quadrilateral, and by joining OA and OB the reactions at A and B are determined in such a way that each lies within its friction cone.

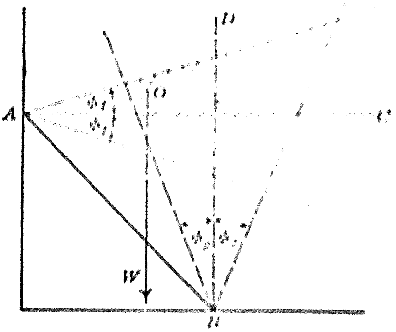


FIG. 150

Whenever the resultant W cuts the quadrilateral, equilibrium is possible in an infinite number of ways, depending on the choice of the point O. Which of all the possible conditions of equilibrium actually exists, it is impossible to determine, since this depends on how the ladder was placed in position, whether revolved around B until it rested at A, or rested first against the wall and then shoved up to A.

As a second example, determine how great the horizontal force F in Fig. 151 must be to start the wedge in against the load W. Beginning at any point A in the vertical surfaces in contact, draw the normal to these surfaces and lay off their friction angle ϕ_1. Since the motion is to be upward, the friction will act downward, and hence the resultant pressure F_1 between the surfaces will be inclined downward as shown. Proceeding to the next frictional surface, B, the reaction F_2 must make an angle ϕ_2 with the normal. Hence, of the three forces F_1, F_2, and W acting on the vertical wedge, the directions of all three are known and the amount of one of them, namely, W. Consequently, the amounts of F_1 and F_2 may be found from a force triangle, as shown in the figure.

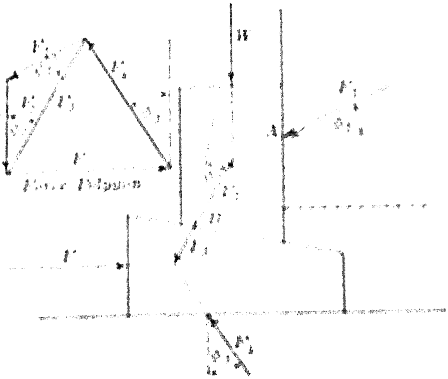


FIG. 151

The horizontal wedge may then be treated in a similar manner, thus determining F_4 and finally F.

As a third example, consider a simple elevator car sliding on a vertical support, as shown in Fig. 152. A load W is carried by the bracket and raised by a cable attached to the upper arm. Since F and W are not in the same line, but form a couple, normal reactions must occur at A and B, forming an equal couple. When motion occurs, these normal reactions create friction. If the motion is upward, the friction will act downward, and hence the reactions at A and B will form an angle ϕ with the normal, as

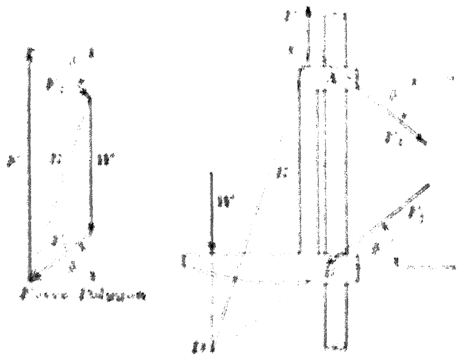


FIG. 152

shown in the figure. If, then, F_2 is combined with F into a single resultant, and similarly F_1 is combined with W, these two resultants must be equal and have the same line of action, CD. The value of F necessary to cause motion may, therefore, be found by first laying off W to scale, and knowing the directions of F_1 and CD, forming a force triangle on these three. Having thus determined CD in amount and direction, and knowing the directions of F_2 and F, their amounts may be found by means of another force triangle, as shown in the figure.

In the solution of the following problems it is only necessary to observe that when motion occurs, the reaction of the plane will make an angle ϕ with the normal to the plane.*

PROBLEMS

262. Show that the maximum inclination α which can be given a plane so that a body on it shall remain at rest is when $\alpha = \phi$.

263. Show that the horizontal pull P necessary to start a body of weight W up a plane of inclination α is given by

$$P = W \tan (\alpha + \phi).$$

264. Show that the horizontal push necessary to start the body down the plane is given by

$$P = W \tan (\phi - \alpha).$$

265. Show that the force parallel to the plane required to start the body up the plane is given by

$$P = W \frac{\sin (\alpha + \phi)}{\cos \phi} = W (\sin \alpha + \mu \cos \phi).$$

266. Find the least force necessary to start a body up a plane of inclination α.

SOLUTION. Lay off AB to scale to represent W (Fig. 153). When motion occurs, the reaction R must form an angle ϕ with the normal to the plane, and hence an angle $\alpha + \phi$ with the vertical. By laying off BC at an angle α with AB and then BD at an angle ϕ with BC, the direction of R is determined. The closing

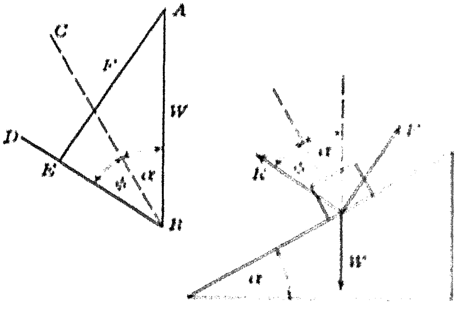


FIG. 153

* Several of the problems in this article are given by Föppl.

side F of the force triangle is then least when it is drawn from A perpendicular to BD. Hence, from the right triangle ABE,

$$F = W \sin(\alpha + \phi).$$

If the body rests on a horizontal plane, $\alpha = 0$. In this case the least force required to produce motion is $F = W \sin \phi$.

267. Show that the least force required to start a body down a plane of inclination α is given by $F = W \sin(\phi - \alpha)$.

268. Find the magnitude of the force acting in any given direction required to start a body up a plane.

SOLUTION. Let θ denote the inclination of the force to the plane (Fig. 154). Then from the force triangle,

$$\frac{F}{W} = \frac{\sin ABC}{\sin ACB} = \frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)}.$$

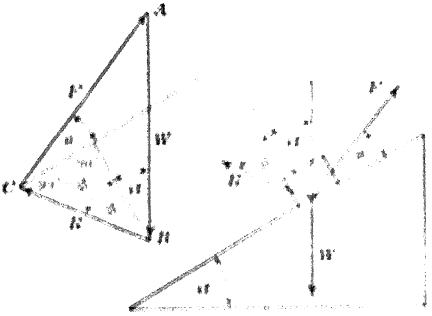


FIG. 154

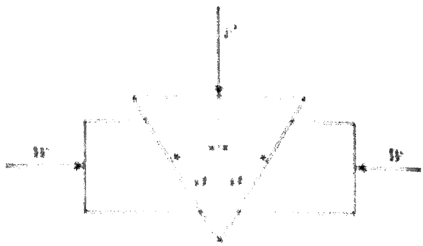


FIG. 155

269. Find the force required to start a double wedge of angle 2α against a resistance W.

SOLUTION. For a single inclined plane of angle α, $F = W \tan(\alpha + \phi)$. Hence for the wedge shown in Fig. 155,

$$F = 2W \tan(\alpha + \phi).$$

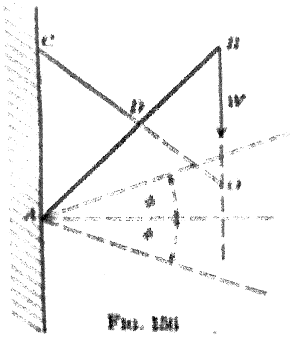


FIG. 156

The wedge will not hold itself in position, but will spring back, if $\alpha > \phi$.

The pull required to withdraw the wedge is, from Prob. 264, equal to

$$F = 2W \tan(\phi - \alpha).$$

270. A bar AB rests with one end A against a vertical wall, and is supported by a cord CD (Fig. 156). What is the condition for equilibrium when a load W is hung from the outer end of the bar?

Solution. Draw a normal to the wall at A and lay off the friction angle ϕ on either side of this normal. Then if CD prolonged intersects W within the area so formed, the bar will be in equilibrium.

271. A prism rests on a horizontal surface. If a horizontal force is applied to the prism near the base, it will cause it to slide; if applied higher up, to tip over. Find the condition in each case.

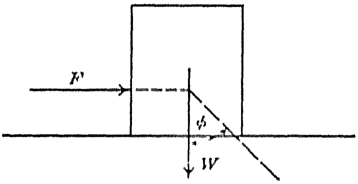


Fig. 157

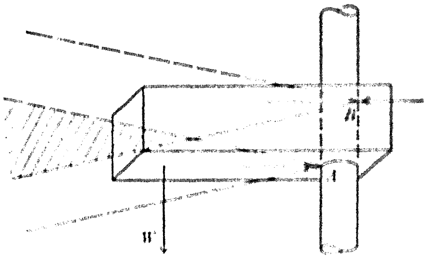


Fig. 158

Solution. Prolong F until it intersects W, and from this point draw the friction angle ϕ (Fig. 157). If the side of this angle intersects the base, the prism will slide; if it falls outside the base, it will tip over.

272. A clamp consists of an arm with a circular hole in one end through which passes a fixed bar (Fig. 158). If a load W is applied to the opposite end of the arm, what is the condition that the clamp shall not slide on the bar?

Solution. When the load W is applied, reactions are introduced at A and B, and hence friction is developed. Draw the normals at A and B and lay off the friction angles on both sides of each. Then if W cuts the shaded area common to both angles, the clamp will remain fixed, since by joining any point on W within the shaded area with A and B, reactions are obtained which lie wholly within the friction angles.

88. Friction Circle. — Consider a cylindrical pivot or journal resting in a bearing which is somewhat loose so that contact between the two is practically along a line (Fig. 159). The looseness is much exaggerated in the figure to illustrate the point in question.

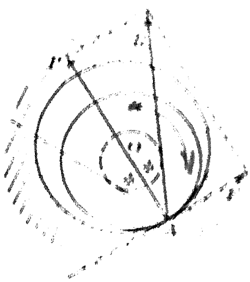


Fig. 159

If the reaction of the pivot on the bearing passes through the center O, the pivot is in equilibrium and there is no friction. Conversely, if there is no friction and the pivot is in equilibrium, the reaction of the bearing must pass through its center. As soon, however, as relative motion occurs between pivot and bearing, friction is de-

veloped, and consequently the reaction no longer passes through the center O, but is inclined to the normal OA at the angle of friction ϕ, as in the case of sliding friction.

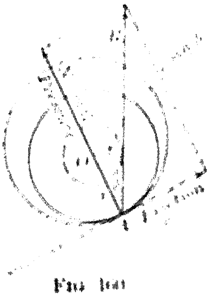


FIG. 160

Since the point of contact A of a rotating shaft or pivot is not usually known, it is convenient to introduce some simple device for determining the direction of the resultant pressure R. For this purpose let r denote the radius of the pivot or journal (Fig. 160). Then the perpendicular distance of R from O is $r \sin \phi$. Hence if a circle is drawn about O as center with radius $r \sin \phi$, the resultant pressure R on the bearing is tangent to this circle. The circle so obtained is called the **friction circle**, and affords a simple practical device for the graphical solution of problems in the kinematics of machinery.

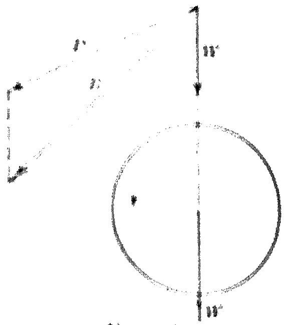


FIG. 161

In the case of a worn bearing or loose fit, the resultant necessarily passes through some point of the surfaces in contact, as in the explanation just given. Suppose, however, that a bearing grips its journal or pin very tightly, so that a great amount of friction is developed. Let W denote the load on the bearing and P the force required to just cause turning (Fig. 161). If the joint is very stiff, the force P required to cause rotation may be so large in comparison with W that their resultant R falls entirely outside the surface of the bearing.

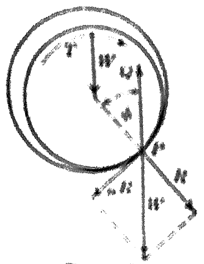


FIG. 162

In the case of a loose or worn bearing this of course cannot happen. In the latter case the effect of the frictional moment T is to cause the axle to ride up on the bearing to a point P such that the angle ϕ is the angle of repose and PQ is equal and parallel to W (Fig. 162). The moment of PQ about O must then equal the frictional moment T.

To obtain a more general demonstration of the friction circle,

let W denote the load on the bearing and P the force required to overcome friction and just produce motion (Fig. 163). Let R denote the resultant of P and W, and resolve this resultant into normal and tangential components. Then if N denotes the normal component, the tangential component or friction will be μN. Consequently we have

$$\frac{AC}{AB} = \frac{\mu N}{N} = \mu = \tan \phi,$$

and hence

$$\measuredangle OBD = \measuredangle ABC = \phi.$$

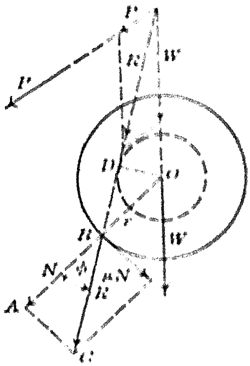


FIG. 163

The perpendicular distance of the resultant R from the center is therefore $OD = r \sin \phi$, and hence if a circle is drawn about O as center with radius $r \sin \phi$, the reaction R will be tangent to this circle.

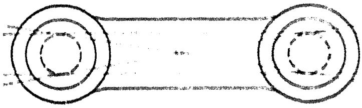


FIG. 164

In mechanisms there frequently occur intermediate rods or links with pins or eyes at each end, connecting them with other machine parts (Fig. 164). From what precedes it is evident that the transfer of force by such a link must take place along one of the four tangents which can be drawn to the two friction circles. Which of these four tangents represents the line of action of the force may be determined by the relative motion at the two ends of the link.

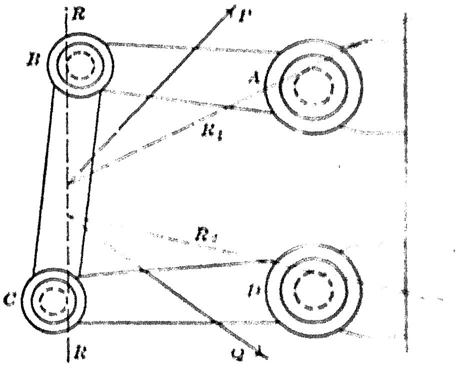


FIG. 165

To illustrate how this is done, consider a linkage of three members, as shown in Fig. 165. Let P denote the driving force acting on the link AB and Q the resistance acting on the link CD. First draw the friction circles for the four joints. Then consider the link BC. For motion in the direction indicated, the line of action

of the force transmitted by this link will be tangent externally to the friction circle at B, and internally at C, as shown in the figure.

A simple way to check the correctness of this statement is to remember that friction tends to decrease the efficiency of a mechanism, which in the present case means that it increases the lever arm of the resistance. Thus, for the upper link AB, by taking moments about A it is evident that the arm of the resistance is increased (or the effectiveness of P decreased) by drawing the tangent on the left of the friction circle, while similarly for the lower link the lever arm of the driving force is decreased by drawing the tangent on the right of the friction circle at C.

To complete the solution, note that the link AB is held in equilibrium by three forces; namely, the driving force P, the reaction R of the link BC, and the reaction at A. The first two of these are known in position and direction, and the third must pass through their point of intersection and be tangent to the friction circle at A. Hence, prolong P until it intersects the line of action R of BC, and from their point of intersection draw a tangent to the friction circle at A, as shown in the figure. This determines the line of action of the reaction R_1 of the pin A. Knowing the direction of these three forces and the amount of one of them, P, the other two may be determined from a force triangle.

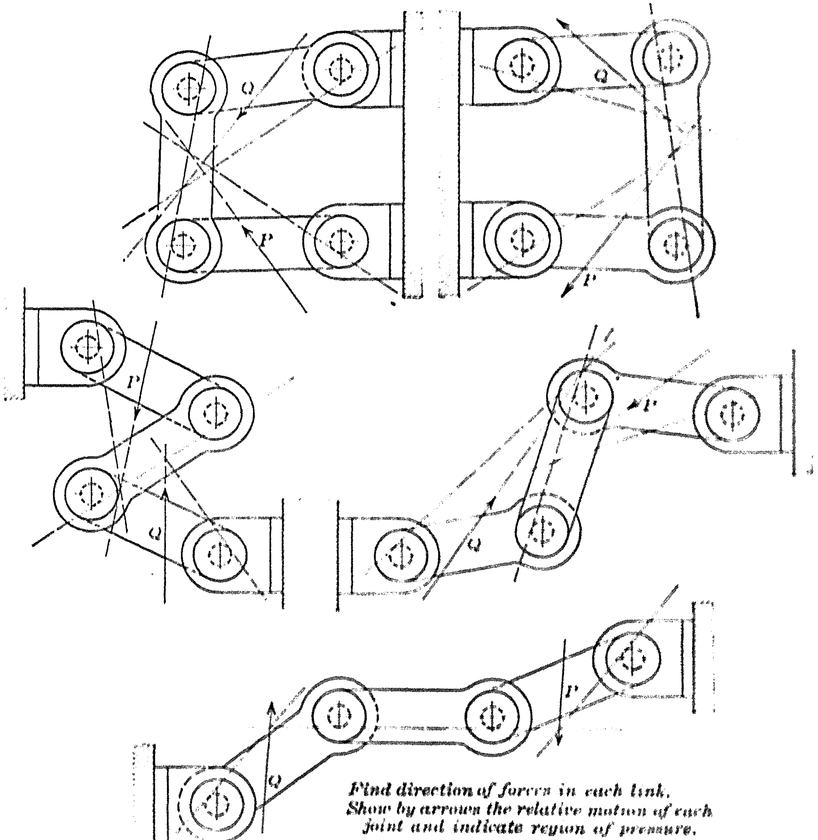
Similarly, by prolonging the line of action of Q until it intersects R, the line of action of the link BC, and from their point of intersection drawing a tangent to the friction circle at D, the direction of the reaction R_2 is determined.

Having determined the amount of R from the previous construction, the value of Q may then be found from a force triangle for R, R_2, and Q.

The following problems further illustrate the practical use of the friction circle.

PROBLEMS

273. Draw the linkages shown in Fig. 166 and determine without further reference to the figure how the various tangents to the friction circles are drawn for the cases illustrated. Then check your results by comparing them with the solutions shown on the figure.



Find direction of forces in each link. Show by arrows the relative motion of each joint and indicate region of pressure.

FIG. 166

NOTE. — A clear understanding of this typical problem is essential to further progress in studying the graphical statics of mechanism in which friction is taken into account. Having once mastered this problem, the solution of other problems is largely a question of detail.

274. Fig. 167 shows a crosshead and connecting rod. Draw the friction circles to scale and determine graphically the relation of the piston pressure P to the resistance Q, using the data given in the table accompanying the cut.

275. Draw the Pittsburgh riveter shown in Fig. 168 to double the scale of the cut, then determine independently of the solution indicated on the figure the directions of the various reactions, and finally draw a complete force polygon for all reactions.

89. Pivot Friction. — A pivot is often so placed that the thrust is applied normal to the end, *i.e.* endways, as in the case of the

Data

Diam. shaft,	8″	μ, journal,	.125
" crank pin,	6″	P	30,000 lb.
" wrist pin,	4″	Q, to be found	
μ, sliding,	.100		

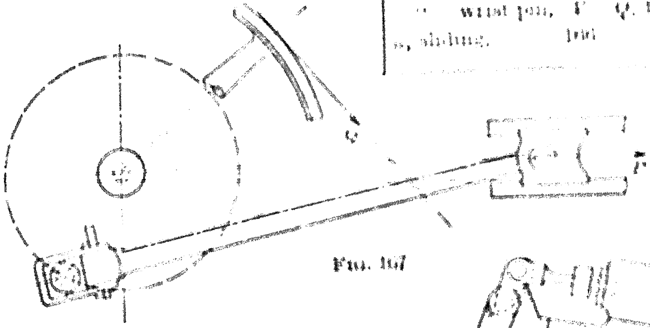


FIG. 167

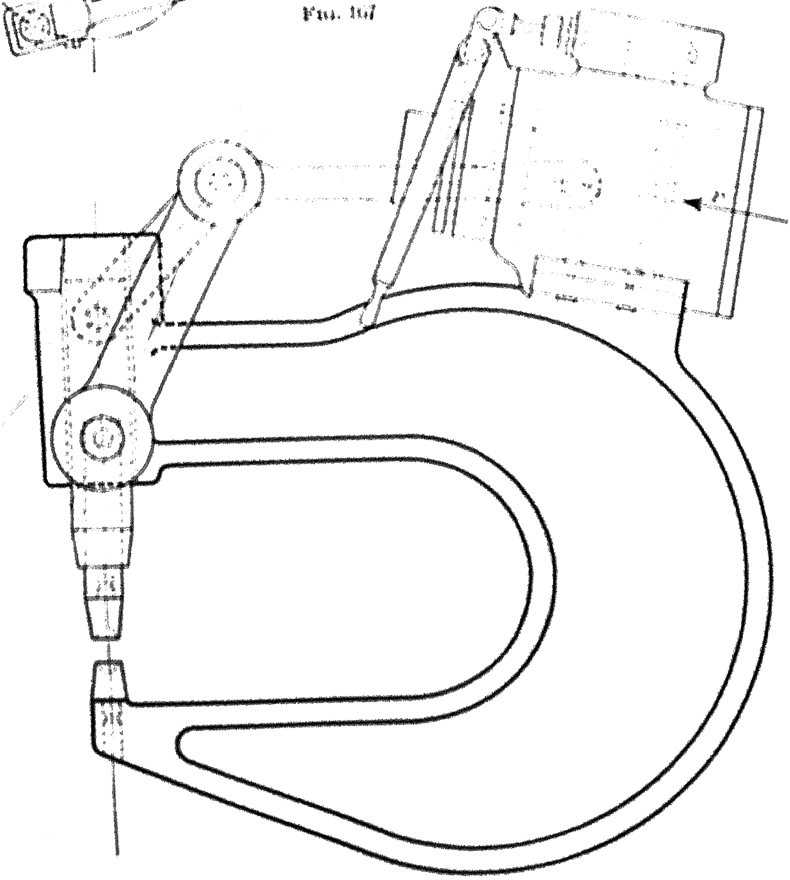


FIG. 168

footstep of a vertical shaft. To determine the frictional moment, and the horsepower absorbed by friction in such a bearing, consider first a frustrated conical pivot, as this case includes all the others.

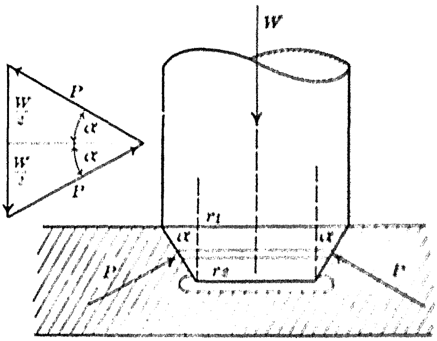


FIG. 169

Let Fig. 169 represent a frustrated conical pivot, and W denote the axial load. Resolve W into two components P, each normal to the bearing surface. Then if α denotes the half angle of the cone,

$$P = \frac{W}{2 \sin \alpha}.$$

Also, if r_1 and r_2 denote the radii of the frustum, the area of half the bearing surface is

$$A = \frac{\pi(r_1^2 - r_2^2)}{2 \sin \alpha}.$$

Therefore since each of the two components P acts over half of the entire bearing surface, the unit pressure p on the bearing is

$$p = \frac{P}{A} = \frac{W}{\pi(r_1^2 - r_2^2)}.$$

It is noteworthy that this unit pressure is independent of the angle of the cone, and is equal to the total thrust divided by the horizontal projection of the bearing area.

For an elementary ring of radius r and width dl, the total normal pressure is $2\,\pi r dl p$. The friction on this annular strip, or ring, is this value multiplied by the coefficient of friction μ. Therefore multiplying the result by the radius r, or distance of the frictional force from the center, the moment of friction is $2\,\pi\mu r^2 dl p$. Inserting the value of p given above, and integrating over the entire surface, the total frictional

moment T is found to be

$$T = \frac{2\mu W}{\sin\alpha(r_1^2 - r_2^2)}\int_{r_2}^{r_1} r^2 dr$$

$$= \frac{2\mu W(r_1^3 - r_2^3)}{3\sin\alpha(r_1^2 - r_2^2)}.$$

Now if N denotes the number of revolutions per minute, the horsepower absorbed by friction is $\frac{2\pi NT}{12 \times 33000}$, which, by inserting the above value of T and reducing, becomes in terms of the diameters

$$\text{h. p. absorbed} = \frac{\mu WN(d_1^3 - d_2^3)}{189000(d_1^2 - d_2^2)\sin\alpha}.$$

Conical Pivot. For a conical pivot of diameter d and half angle α (Fig. 170), the values of the frictional moment and the

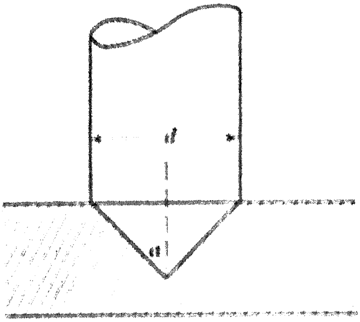


Fig. 170

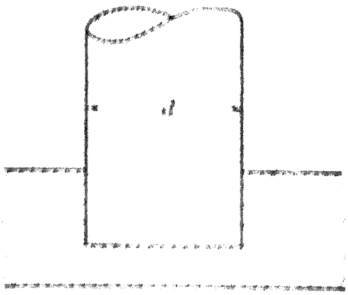


Fig. 171

horsepower absorbed may be obtained from the above by putting $d_2 = 0$ and $d_1 = d$, in which case

$$T = \frac{\mu WNd}{3\sin\alpha},$$

$$\text{h. p. absorbed} = \frac{\mu WNd}{189000\sin\alpha}.$$

Flat Pivot. Here $d_2 = 0$, $d_1 = d$, and $\alpha = 90°$ (Fig. 171).

Hence
$$T = \frac{\mu Wd}{3},$$

$$\text{h. p. absorbed} = \frac{\mu WNd}{189000}.$$

Hollow Flat Pivot or Collar Bearing. In this case $a = 0$ (Fig. 172). Hence

$$T = \frac{\mu W(d_1^3 - d_2^3)}{3(d_1^2 - d_2^2)},$$

$$\text{h. p. absorbed} = \frac{\mu WN(d_1^3 - d_2^3)}{189000(d_1^2 - d_2^2)}.$$

Cylindrical Journal. In the case of a shaft or journal revolving in a plain bearing, the frictional force is everywhere tangential to the cylinder, and is of amount $F = \mu W$, where W denotes the total load on the bearing. If the shaft is of r ft. radius and makes N rotations per minute, the work done in one rotation is $2\pi rFN$ ft.-lb., and the power absorbed by friction is

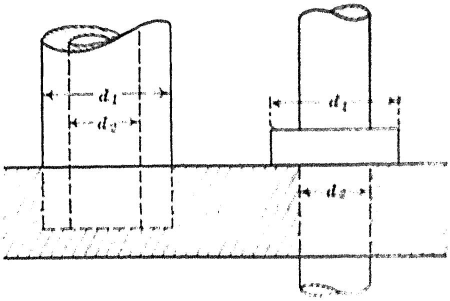


FIG. 172

$$\text{h. p. absorbed} = \frac{2\pi r\mu WN}{33000}.$$

PROBLEMS

276. Find the h. p. required for a shaft 9 in. in diameter, carrying a load of 15 T., and running at 90 r. p. m., assuming that the coefficient of friction is .15.

277. Find the h. p. absorbed by a footstep bearing, 6 in. in diameter, carrying a load of 3 T. and running at a speed of 150 r. p. m. if the coefficient of friction is .02.

278. Assuming that the energy absorbed (or wasted) between the end of a flat pivot and its bearing is proportional at each point to the speed and pressure jointly, show that the total energy absorbed by the bearing is $\frac{2}{3} Wr\mu\omega$, where ω denotes the angular velocity in radians per second.

279. A journal is 9 in. long, 6 in. in diameter, and carries a load of 4 T. at a speed of 90 r. p. m. If the coefficient of friction is .02, what h. p. is wasted and what is the efficiency of the bearing?

90. Screw Friction; Square-threaded Screw. A screw thread or helix is simply an inclined plane wrapped around a cylinder. Let h denote the pitch of the screw; that is to say, the distance a

point on the thread rises in one complete revolution of the screw (Fig. 173). Consider first a square-threaded screw, and take for its diameter the mean diameter of the threads (*i.e.* average of diameters at base of thread and outside of thread). A square-threaded screw is then in effect a narrow inclined plane of base πd and height h. Hence the force

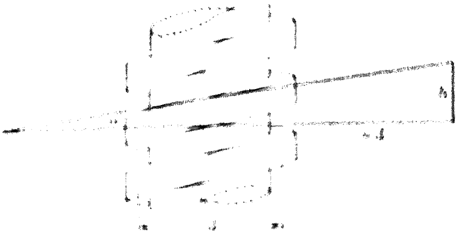


FIG. 173

P required at the mean diameter d to raise a weight W is, from Prob. 258,

$$P = W \tan(\alpha + \phi).$$

If the force is applied by means of a wrench or lever of length l (Fig. 174), the force F required to raise the load W is given by $Fl = P\frac{d}{2}$, whence

$$F = \frac{Wd}{2l}\tan(\alpha + \phi).$$

Square-threaded Bolt and Nut. In addition to the friction on the threads, the friction on the face of the nut or thrust collar must ordinarily be taken into account. For an ordinary nut the mean radius may be assumed as $1\frac{1}{2}$ the mean radius of the threads. Under this assumption the above expression becomes simply

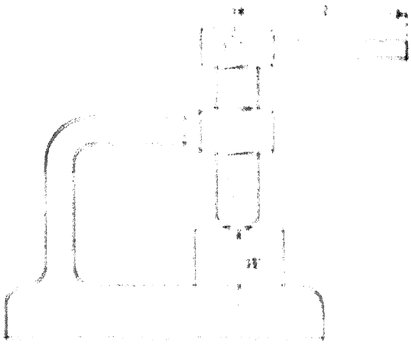


FIG. 174

$$P = W \tan(\alpha + 2.5\,\phi),$$

and

$$F = \frac{Wd}{2l}\tan(\alpha + 2.5\,\phi).$$

Thrust Collar. For a thrust collar the mean radius may be assumed equal to that of the threads; in which case

$$P = W \tan(\alpha + 2\,\phi),$$

and

$$F = \frac{Wd}{2l}\tan(\alpha + 2\,\phi).$$

V-*threaded Screw.* For a triangular, or V-shaped, thread the normal pressure W' is greater than W, as shown in the force triangle in Fig. 175, the relation being

$$\frac{W}{W'} = \cos\frac{\theta}{2}.$$

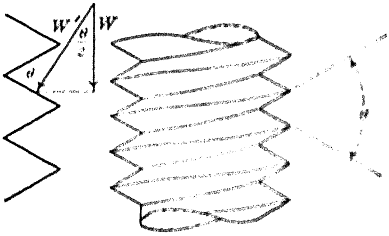


FIG. 175

For the Whitworth thread $\theta = 55°$, and for the Sellers thread $\theta = 60°$. As an average value we may therefore assume

$$W' = 1.14\ W,$$

in which case $P = W \tan(\alpha + 1.14\ \phi)$, and hence

$$F = \frac{Wd}{2\ l} \tan(\alpha + 1.14\ \phi).$$

V-*threaded Bolt and Nut.* For a V-threaded bolt and nut we have, by the same reasoning as above, $P = W \tan(\alpha + 2.64\ \phi)$, whence

$$F = \frac{Wd}{2\ l} \tan(\alpha + 2.64\ \phi).$$

PROBLEMS

280. A square-threaded bolt has 10 threads to the inch, mean diameter of threads 1.5 in., average outside diameter of nut 3 in., inside diameter of hole in washer under nut 1.75 in. What pull can be exerted on the end of a 3 ft. wrench in order that the stress in the bolt shall not exceed 40,000 lb.? Coefficient of friction assumed as .15.

SOLUTION. Problems in screw friction may be solved by the method given above, or by applying the principle of work. The above problem will be solved by both methods for the sake of comparison.

Consider first the work done by all the forces involved in one revolution of the screw. Then if F denotes the force applied at the end of the wrench, the work done in one revolution is

$$\text{work of wrench} = F \times 2\,\pi\,36.$$

Since the total load is limited to 40,000 lb.,

$$\text{work done on threads} = 40{,}000 \times .15 \times \pi \times 1.5.$$

Similarly

$$\text{work done on face of nut} = 40{,}000 \times .15 \times \pi \times 2.375,$$

and

$$\text{work of lifting} = 40{,}000 \times .1.$$

The equation of work therefore becomes

$$F \cdot 2\pi \cdot 36 = 40{,}000(.15\pi 1.5 + .15\pi 2.375 + .1),$$

whence

$$F = 340 \text{ lb.}$$

Otherwise, from the formula obtained above for the friction of a square-threaded screw with nut, namely,

$$F = \frac{Wd}{2l}\tan(\alpha + 2.5\phi),$$

we have in the present case $\tan\alpha = \frac{.1}{1.5\pi}$ and $\tan\phi = .15$, whence $\alpha = 1°\,13'$ and $\phi = 8°\,32'$. Also $W = 40{,}000$ lb., $d = 1.5$ in., and $l = 36$ in. Substituting these values in the formula, the result is

$$F = 328 \text{ lb.}$$

281. A screw of mean diameter $2\frac{1}{4}$ in. has 4 threads to the inch. Assuming the coefficient of friction as .08, calculate its efficiency.

282. A 1-in. bolt has 8 threads to the inch of mean diameter .93 in. Assuming the coefficient of friction as .15, find the torque necessary to raise an axial load of 3 T.

283. A square-threaded screw has 5 threads to the inch of mean diameter 1.2 in. What axial force will it exert when acted on by a torque of 40 ft.-lb.?

91. Efficiency and Reversed Efficiency. In all machines the work put into the machine is partly expended in overcoming friction, and is therefore greater than that obtained from the machine. The efficiency of a machine is defined as

$$\text{efficiency} = \frac{\text{output}}{\text{input}}.$$

Since the denominator of this fraction is always greater than the numerator, the efficiency of a mechanism is always less than unity.

Let the work put into a machine be denoted by W, that expended in overcoming friction by W_f, and that obtained from the machine as useful work by W_u. Then

$$W = W_f + W_u,$$

and consequently the efficiency e is

$$e = \frac{W_u}{W} = \frac{W_u}{W_u + W_f}.$$

Now suppose that the machine is reversed, so that the original resistance becomes the driver, and the original driver becomes

the resistance. Since friction always acts in opposition to motion, the condition for equilibrium now becomes

$$W = W_U - W_F.$$

The efficiency of the reversed mechanism is called the **reversed efficiency.** Its value is

$$e_r = \frac{W}{W_U} = \frac{W_U - W_F}{W_U}.$$

The reversed efficiency must also be less than unity, and becomes negative when the frictional resistance of the machine is greater than the useful work done. In this case the machine will be self-sustaining; that is to say, it will not run backward. Screw jacks and differential pulleys are examples of such self-sustaining mechanisms. The least frictional resistance required to make a machine self-sustaining is when

$$W_U = W_F.$$

Substituting this relation in the expression for efficiency of forward motion, it becomes

$$e = \frac{W_U}{2\,W_U} = \frac{1}{2}.$$

Hence the efficiency of a self-sustaining mechanism can never exceed 50 per cent.

92. Efficiency of Various Machine Elements. — Inclined Plane. The efficiency of an inclined plane may be found at once from the definition

$$e = \frac{\text{output}}{\text{input}} = \frac{\text{work assuming friction to be zero}}{\text{total work including friction}}.$$

The most general case is when the force is inclined at an angle θ to the plane. In this case, from Prob. 268,

$$F = W\frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)}.$$

If friction is neglected, $\phi = 0$ and $F = W\dfrac{\sin\alpha}{\cos\theta}$. Hence

$$e = \frac{W\dfrac{\sin\alpha}{\cos\theta}}{W\dfrac{\sin(\alpha + \theta)}{\cos(\theta - \phi)}} = \frac{\sin\alpha\cos(\theta - \phi)}{\cos\theta\sin(\alpha + \phi)}.$$

If the force F acts parallel to the plane, $\theta = 0$ and

$$e = \frac{\sin\alpha\cos\phi}{\sin(\alpha + \phi)} \qquad \textit{(Force parallel to plane).}$$

If the force F acts horizontally, $\theta = -\alpha$, and

$$e = \frac{\sin\alpha\cos(\alpha + \phi)}{\cos\alpha\sin(\alpha + \phi)} = \frac{\tan\alpha}{\tan(\alpha + \phi)} \qquad \textit{(Force horizontal).}$$

Wedge. A wedge is simply a double inclined plane, and its efficiency is therefore

$$e = \frac{\tan\alpha}{\tan(\alpha + \phi)}.$$

The reversed efficiency of a wedge, that is, when it is being withdrawn, is

$$e_r = \frac{\tan(\alpha - \phi)}{\tan\alpha}.$$

Square-threaded Screw. The efficiency of a square-threaded screw is the same as for an inclined plane; namely,

$$e = \frac{\tan\alpha}{\tan(\alpha + \phi)} \qquad \textit{(Square-threaded screw).}$$

For a screw thread and thrust collar, assuming that the average diameter of the collar is the same as for the thread,

$$e = \frac{\tan\alpha}{\tan(\alpha + 2\phi)} \qquad \textit{(Square-threaded bolt and thrust collar).}$$

For a screw and nut, assuming that the average diameter of the nut is $1\frac{1}{2}$ that of the thread,

$$e = \frac{\tan\alpha}{\tan(\alpha + 2.5\phi)} \qquad \textit{(Square-threaded bolt and nut).}$$

If the screw is reversible, that is, if the nut will drive the thread, as in the case of the automatic screw driver in common use, the reversed efficiency is

$$e_r = \frac{\tan(\alpha - \phi)}{\tan\alpha} \qquad \textit{(Square threaded bolt and nut).}$$

V-threaded Screw. Referring to Art. 90, the efficiency of a V-threaded screw is found to be

$$e = \frac{\tan \alpha}{\tan(\alpha + 1.14\,\phi)} \qquad (\textit{V-threaded screw}),$$

and for a V-threaded bolt and nut,

$$e = \frac{\tan \alpha}{\tan(\alpha + 2.64\,\phi)} \qquad (\textit{V-threaded bolt and nut}).$$

The following table gives approximate values of the efficiencies of screws for various inclinations or pitches.

Inclination of Thread, α $\left(\tan \alpha = \frac{\text{Pitch}}{\pi \text{ Diam.}}\right)$	Efficiency in Per Cent $\mu = .15$ Including Nut and Washer, or Thrust Collar	
	Square Thread	V Thread
2°	11	8
3°	14	12
4°	17	16
5°	21	20
10°	36	29
20°	48	42

Toothed Gearing. The efficiency of gears depends on the finish and form of teeth, lubrication, etc. The following empirical formulas are due to Goodman, and are very closely approximate.

For one pair of machine-cut toothed gears, including the friction of the axle,

$$e = \left(96 - \frac{40}{n}\right) \text{per cent},$$

where n is the number of teeth in the smaller wheel.

For rough, unfinished gears

$$e = \left(90 - \frac{40}{n}\right) \text{per cent}.$$

For a train of gears, the efficiency of the whole train is equal to the product of the efficiencies of the several pairs.

The following table gives approximate values of the mechanical efficiencies of various machine elements and prime movers.

Machine	Mechanical Efficiency in Per Cent
Simple pulley	90–95
Pulley block of n sheaves	e^n, where e denotes efficiency of one sheave
Simple lever, two pin joints	94–97
Simple lever, knife edges	100
Weston differential pulley	20–40
Steam hoist	50–70
Steam cranes	60–70
Best stationary steam engines	80–90
Locomotives	65–75
Hydraulic windlass	60–80
Hydraulic jack	80–90
Hydraulic turbine	70–80
Undershot water wheel	30–40
Overshot water wheel	60–80
Poncelet water wheel	65–75
Pelton water wheel	75–85
Gas engine	75–85
Steam turbine only	90–95
Steam turbine generating unit	85–93
Electric motor	85–90
Dynamo or generator	90–95
Traveling cranes	30–50

93. Rolling Friction. — A rolling wheel carrying a load experiences resistance to motion, due to the irregularities or deflection of the surface over which it rolls. This rolling resistance, however, is in general much smaller than sliding friction would be for the same load. Axle friction must also be added to the rolling resistance, although it is usually small in comparison with the latter.

When a wheel is at rest, the reaction of the bearing passes through the center of the axle and the point of contact of the wheel with the ground. As soon as the wheel begins to move, however, the axle pressure becomes tangent to the friction circle

drawn for the axle, as indicated in Fig. 176. The vertical component of the reaction F is equal to the load on the wheel W, and the horizontal component is the axle friction. Calling this horizontal component H, we have

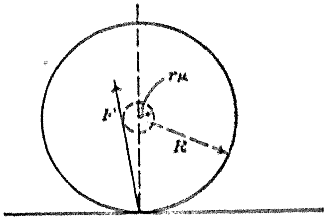


FIG. 176

$$\frac{H}{W}=\frac{r\mu}{R}, \text{ or } H=\frac{Wr\mu}{R}.$$

Since H is inversely proportional to the radius of the wheel R, the advantage obtained by using large wheels is obvious. Also, if the wheels of a truck are not all of the same size, it is evident that most of the load should be carried over the larger wheels.

Consider next the rolling resistance between wheel and roadbed. When a wheel rolls on yielding material, it forms a rut. If the roadbed is inelastic, as in the case of dirt and macadam roads, this rut remains after the wheel has passed. If the roadway is composed of an elastic material like wood, asphalt, or steel rails, the material is compressed in front of the wheel, but regains its form more or less completely after the wheel has passed, so that no rut is formed.

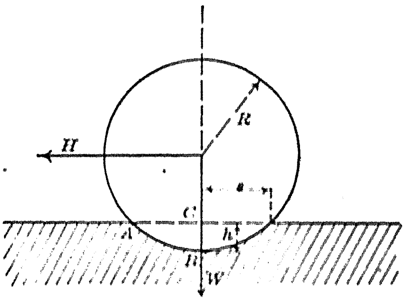


FIG. 177

In either case the resistance experienced is equivalent to constantly mounting an obstacle in front of the wheel. Let H denote the horizontal pull necessary to overcome this resistance, and W the load on the wheel. Then taking moments about A, Fig. 177,

$$H(R-h)=Ws,$$

or, neglecting h in comparison with R,

$$H=\frac{Ws}{R}, \text{ approximately.}$$

This formula is only approximate at best, since it assumes that the resultant reaction of the wheel and the ground passes through

A, whereas, in fact, it passes through the center of gravity of the compressed area ABC. The incompleteness of the data on this subject, however, does not warrant the derivation of a more rigorous formula at present.

Values of s* for use in this formula have been empirically determined as follows:

MATERIAL	s, in inches
Iron or steel wheels on iron or steel rails	.007 .015
Iron or steel wheels on wood	.06 .10
Iron or steel wheels on macadam	.05 .20
Iron or steel wheels on earth	3.00 5.00
Pneumatic tires on good road or asphalt	.02 .022
Solid rubber tires on asphalt	.04
Pneumatic tires in mud	.04 .06
Solid rubber tires in mud	.09 .11

Morin gives the value of s for wagons on soft soil as .065 in., and on hard roads as .02 in. He also found that on hard, dry roads, with wheels 44 in. front and 54 in. rear, and either 1½-in. or 3-in. tires, that the traction was 114 lb. / T. On wood block pavement the traction with 1½-in. tires was 28 lb. / T., and with 6-in. tires, 38 lb. / T.

According to Baker,† axle friction varies from .012 to .02 of the load for good lubrication. For ordinary American carriages the traction necessary to overcome axle friction is from 3 to 3½ lb. / T., and for ordinary wagons from 3½ to 4½ lb. / T. The total traction on macadam roads for wheels 50 in. in diameter was found to be 57 lb. / T.; for wheels 30 in. in diameter, 61 lb. / T.; and for wheels 26 in. in diameter, 70 lb. / T.

94. Rope and Belt Friction. — Consider a rope or belt passed around a fixed cylindrical body, such as a post or tree trunk. Let the pull on the ends be denoted by F_1 and F_2 (Fig. 178). Then the problem is to determine how much F_2 must exceed F_1 before the limiting friction is reached and slipping begins.

* This quantity s is sometimes incorrectly called the "coefficient of rolling friction." It is not, properly speaking, a coefficient at all, since it is measured in inches.

† *Eng. News*, March 6, 1902.

Draw any two consecutive radii OA and OB. Then if the tension at any point A is denoted by F, the tension at a neighboring point B will be $F + dF$. Now let a force triangle be formed on the three forces F, $F + dF$, and the reaction R, the latter being tangent to the friction circle drawn about O, *i.e.* R forms an angle ϕ with the normal N. Expressing $d\theta$ in circular measure, we have from the force triangle, $N = Fd\theta$. Hence the friction dF on the segment AB becomes

$$dF = N\mu = \mu F d\theta.$$

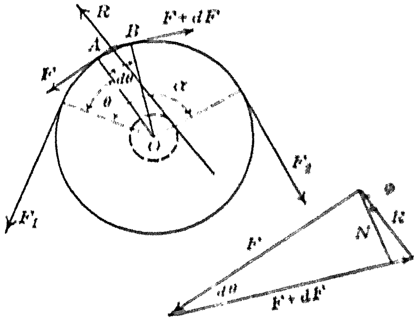


FIG. 178

Writing this expression in the form

$$\frac{dF}{F} = \mu d\theta,$$

and integrating over the entire arc of contact θ, we have

$$\int_{F_1}^{F_2} \frac{dF}{F} = \int_0^\theta \mu d\theta,$$

whence

$$\log_e \frac{F_2}{F_1} = \mu\theta.$$

Expressing this in terms of the inverse, or exponential, function, this relation may also be written

$$F_2 = F_1 e^{\mu\theta}.$$

This formula can be changed to a form more convenient for computation by changing from the Napierian to the common system of logarithms, and writing $\theta = 2\pi n$, where n is the number of laps which the belt, or rope, makes around the cylinder. Making these substitutions, we have $\log_{10} F_2 - \log_{10} F_1 = .4343\, \mu \times 2\pi n$, or

$$\log_{10} F_2 = \log_{10} F_1 + 2.7288\, \mu n.$$

This formula may also be used for calculating the friction of belts on pulleys. In order that the belt shall not slip on the pulley, F_2 cannot exceed the value given by the above formula.

The difference between F_2 and F_1 is, then, the driving force acting on the circumference of the pulley.

There are also in general use tables and empirical formulas giving the approximate transmitting power of single and double leather belts, based on actual results obtained in practical operation. It has thus been found that under average conditions a single leather belt running at 600 ft./min. will transmit 1 h. p. for each inch of width, and that a double leather belt running 400 ft./min. will also transmit 1 h.p. per inch of width, these values being based on 180° arc of contact. From these results the following empirical formula for calculating the horsepower transmitted has been deduced:

$$\text{h.p.} = \frac{wv}{k},$$

where w = width of belt in inches,
v = speed in ft./min.,
k = empirical constant, as given below.

Kind of Belt	Weight, Oz. per Ft	k
Single	16	625
Light double	24	417
Medium double	28	357
Standard double	33	303
Three ply	45	222

This table is based on 180° arc of contact. When the arc of contact is less than 180°, the transmitting power of the belt is reduced, and varies with the arc of contact as shown by the percentages in the following table:

Arc of Contact, Degrees	Per Cent of Transmitting Power Based on 180° Arc of Contact
180	100
170	97
160	94
150	91
140	87
130	83
120	79
110	75
100	65

To illustrate the use of the formula and tables, let it be required to determine how many horsepower can be transmitted by a 7-in. light double leather belt on a 48-in. pulley at 300 r. p. m. with an arc of contact of 150°.

Here the speed in ft./min. is $v = 4\,\pi\,300$. Hence,

$$\text{h. p. for } 180° \text{ lap} = \frac{4\,\pi\,300 \times 7}{417} = 63.3.$$

Reducing this value to 150° arc of contact by multiplying by 91 per cent as given above, we have finally

$$\text{h. p. for } 150° \text{ lap} = 63.3 \times .91 = 57.6.$$

The centrifugal force developed by a belt running at high speed materially increases its tension. For the low speeds which ordinarily occur this addition to the tension is negligible, but for very high speeds it must be taken into account.

Thus, for a speed of a mile a minute, the centrifugal force developed adds about 100 per cent to the tension. The method of taking this into account is explained in the following paragraph on rope drives.

Rope Drives. In rope drives the friction is increased by using a grooved pulley with a groove smaller than the rope, as shown in Fig. 179, so that the rope cannot bottom in the groove. The normal pressure is thereby increased in the ratio $\dfrac{1}{\sin\frac{\theta}{2}}$, and consequently the friction is also increased in the same ratio. The angle θ is usually about 45°, for which $\dfrac{1}{\sin\frac{\theta}{2}} = 2.6$. The friction may then be computed as for an ordinary belt by simply using $2.6\,\mu$ for the coefficient of friction instead of μ.

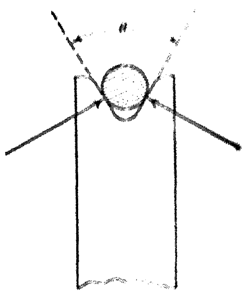


FIG. 179

For a rope on a cast-iron sheave, $\mu = .3$, and hence $2.6\,\mu = .78$. Consequently, for a lap of 180° we have $\theta = \pi$ and $e^{\mu\theta} = 11.6$, whence

$$F_2 = 11.6\,F_1.$$

The working strength in pounds for hemp rope may be taken as $100\,d^2$, where d is the diameter of the rope in inches, which is equivalent to using a factor of safety of about 80. Now the horsepower transmitted by a rope drive is

$$\text{h. p.} = \frac{(F_2 - F_1)v}{33000},$$

where v is the speed in ft./min. Hence, substituting the above values in this formula, namely, $F_2 = 100\,d^2$ and $F_1 = \dfrac{F_2}{11.6}$, we have

$$\text{h. p.} = \frac{d^2 v}{455},$$

where d = diameter of rope in inches,

v = speed in ft./min.

In the case of a rope or belt drive running at high speed, the effective tension is diminished by the centrifugal force. The general expression for centrifugal force is

$$C = \frac{wv^2}{gR},$$

where in this case C may be considered as a radial pressure between the rope or belt and pulley per foot of length. Hence if F denotes the tension in the rope or belt necessary to equilibrate this radial pressure, we have

$$2\,F = CD,$$

where D denotes the diameter of the pulley. Consequently the centrifugal tension F becomes

$$F = \frac{CD}{2} = \frac{wv^2 D}{2\,gR},$$

or, since $2\,R = D$, this becomes

$$F = \frac{wv^2}{g},$$

where w = weight of one foot of rope or belt,

v = speed in ft./sec.,

$g = 32.2$ ft./sec.^{2}.

For a rope drive it has been found by experiment that the adhesion of the rope to the sheave is sufficient to prevent slippage when the initial tension on the driving side is twice that on the slack side; that is, if F denotes the effective tension, namely,

$$F = F_2 - F_1,$$

when the adhesive tension is $\frac{1}{2} F$. Consequently the total effective tension becomes

$$F = F_2 - T - \tfrac{1}{2} F,$$

or,
$$F = \tfrac{2}{3}(F_2 - T).$$

Since
$$\text{h. p.} = \frac{\text{force} \times \text{speed}}{550},$$

we have therefore
$$\text{h. p.} = \frac{\frac{2}{3}\left(F_2 - \frac{wv^2}{g}\right)v}{550},$$

where F_2 = maximum allowable tension.

For a constant tension the power transmitted is evidently directly proportional to the speed, whereas the centrifugal force decreases the effective tension in proportion to the square of the speed. There is a speed therefore at which the power transmitted is a maximum, and such that an increase in speed beyond this value is accompanied by a decrease in power. This is illustrated by the diagram in Fig. 180, the data from which the diagram was plotted being given in the following table.

This table was calculated from the formula for horsepower just derived, taking for the values of the constants the numerical values in actual commercial use; namely,

$$\text{weight per foot} = w = .34\, d^2,$$

$$\text{breaking strength} = 7000\, d^2,$$

$$\text{maximum allowable tension} = F_2 = \frac{7000\, d^2}{35} = 200\, d^2.$$

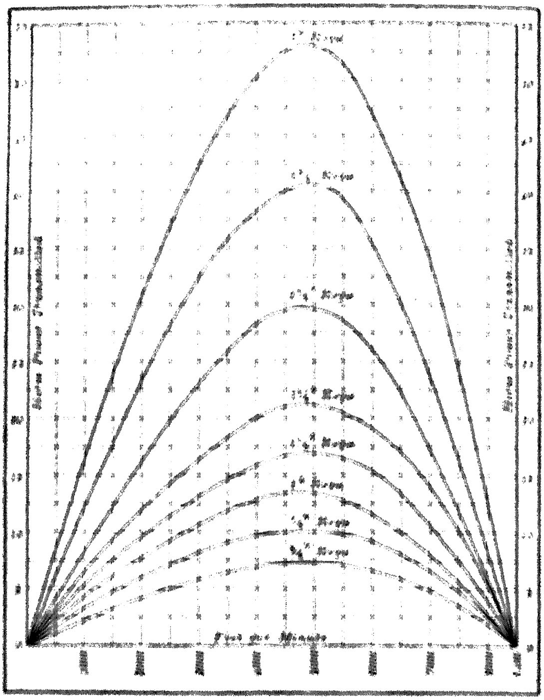


Fig. 180

HORSEPOWER, MANILA ROPE

Diameter of Rope	Velocity, Feet per Minute										
	1000	1500	2000	2500	3000	3500	4000	4500	5000	5500	6000
¾	2.3	3.3	4.3	5.2	6.0	6.6	7.2	7.3	7.4	7.3	6.9
⅞	3.0	4.5	5.9	7.0	8.2	9.0	9.6	9.8	10.0	9.6	9.0
1	4.0	5.9	7.7	9.2	10.6	11.8	12.7	12.9	13.0	12.7	12.0
1⅛	5.0	7.5	9.7	11.6	13.5	14.9	16.0	16.3	16.7	16.5	15.3
1¼	6.3	9.1	12.0	14.3	16.7	18.5	20.0	20.2	20.7	20.1	18.9
1⅜	7.5	10.8	14.4	17.4	20.0	22.1	23.7	24.5	24.6	24.0	22.3
1½	9.0	13.5	17.4	20.7	23.9	26.3	28.7	29.9	29.5	28.6	26.7
1⅝	10.5	15.5	20.1	24.3	27.9	30.8	32.9	34.1	34.3	33.3	31.0
1¾	12.3	18.0	23.6	28.2	32.7	36.4	38.5	39.4	40.5	38.7	36.0
2	16.0	23.2	30.6	36.8	42.5	46.7	50.0	51.7	52.8	50.6	47.3
2¼	20.0	29.6	38.6	46.6	53.6	59.2	63.6	65.8	66.3	64.4	60.3
2½	25.0	36.6	47.7	57.5	66.0	71.2	78.0	80.0	81.0	79.0	73.8

PROBLEMS

284. What size of double belt will be required to transmit approximately 280 h. p. with a flywheel 10 ft. in diameter running at 120 r. p. m., assuming the arc of contact to be 180°?

285. A pulley 5 ft. in diameter transmits 25 h. p. at 140 r. p. m., the lap being 150° and the coefficient of friction .3. Find the tension in the belt.

HINT. — Let T_1 and T_2 denote the tensions in the two sides of the belt. Then from the h. p. transmitted, the size of the pulley and the speed, we may find $T_1 - T_2$. Likewise from the lap and the coefficient of friction we may find $\frac{T_1}{T_2}$. Solving these two equations simultaneously, T_1 and T_2 are obtained.

286. 30 h. p. is being transmitted by a belt running at 4000 ft./min. Assuming the maximum working stress in the belt as 330 lb./in.², how wide must it be if it is $\frac{1}{4}$ in. thick?

287. A rope drive carrying 20 ropes is 14 ft. in diameter and transmits 500 h. p. at 100 r. p. m. Taking $\mu = .3$ and the angle of contact 180°, find the tensions on both sides of the pulley.

288. A rope is wrapped three times around a post and supports a weight of 200 lbs. If the coefficient of friction is .3, find the least force necessary to raise the weight, and also the force required to just prevent it from slipping down.

289. A chain is wrapped twice around an iron drum, and a pull of 100 lb. just supports 50 T. Find the coefficient of friction.

290. A barrel of flour weighing 196 lb. is supported by a rope passing over a round beam. How great a pull will it take to raise it and to lower it, the coefficient of friction being assumed as .5?

291. How many turns must be taken around the drum of an electric capstan in order to exert a pull of $2\frac{1}{2}$ T. on a loaded freight car if a man exerts a pull of 100 lb. on the other end of the rope? Assume the coefficient of friction as .3.

95. Rigidity of Ropes. — No rope is perfectly flexible, but offers more or less resistance to bending. This resistance to bending is greater the more sharply the rope is bent; that is, the smaller its radius of curvature. In the case of wire rope this resistance is partly elastic; that is to say, the rope tends to straighten itself after being bent, so that the work done in bending it is not entirely lost. For hemp rope, however, the resistance is chiefly plastic, and the work done in bending it is lost in friction between the fibers.

The greater [illegible] pressure [illegible] in comparison with the [illegible] due to the difference in tension [illegible] its rigidity and bend [illegible]. Let F denote the driving tension and F_1 the tension [illegible]. Then

$$F_1 = eF,$$

where e denotes the efficiency of the pulley, neglecting axle friction. It is impossible to give any definite numerical value to e, and it is best to determine it experimentally for each individual case. When this is impracticable, the following empirical formula may be used to give an approximate value of the bending resistance of hemp or manila rope, [illegible]:

$$F' = \frac{d^2 F}{R}$$

where F = tension in [illegible] side,

$F + F'$ = tension in [illegible] side,

d = diameter of rope in inches,

R = radius of sheave in inches.

The total pull on the driving side in terms of the load F is then

$$\text{Tension in driving side} = F\left(1 + \frac{d^2}{R}\right)$$

If a rope passes over n pulleys, as in the case of a block with n sheaves, the tension in the last rope is given by

$$F_n = e^n F$$

The efficiency therefore decreases rapidly with the number of sheaves.

Let W denote the load being lifted by a tackle of n sheaves, and F the driving tension. Then from the condition

$$\Sigma \text{ vertical forces} = 0$$

we have
$$eF + e^2F + e^3F + \cdots + e^nF = W$$

The summation of this power series, or geometrical progression, gives

$$W = F\frac{e(e^n - 1)}{e - 1}$$

If $e = 1$, then $W = nF$. Hence the efficiency e' of the entire tackle is

$$e' = \frac{F\dfrac{e^{n+1} - e}{e - 1}}{nF} = \frac{e^{n+1} - e}{n(e - 1)}.$$

Since the flexibility of ropes decreases as their size increases, there is a limit to the size which can be efficiently used for rope driving. This maximum diameter is about 2 in. For ordinary rope drives, $1\frac{1}{2}$ in. rope is commonly used. The diameter of the smallest pulley over which the rope passes must then be at least 30 times the diameter of the rope.*

PROBLEMS

292. A rope 1 in. in diameter passes over a sheave 2 ft. in diameter under a tension of 800 lb. What force is necessary to overcome the stiffness of the rope?

293. How much larger would the sheave have to be to reduce the force necessary to overcome the rigidity of the rope in the preceding problem by 50 per cent.?

294. A tackle of two blocks, each containing three sheaves 8 in. in diameter, has a $\frac{3}{4}$-in. rope running through it. What pull will be required to raise a weight of one ton?

96. Antifriction Wheels.

96. Antifriction Wheels. — A device for reducing axle friction consists in mounting the axle on antifriction wheels, as shown in

* Ropes used for power transmission are made of fibers of selected hemp, Italian hemp being stronger than Russian hemp.

A yarn is a fiber laid left handed, and a strand consists of yarns laid left handed. A hawser has three strands laid right handed, and a cable consists of three hawsers laid left handed.

To keep ropes in a flexible condition the best dressing is castor oil. For new cotton ropes tallow and wax may be used.

The approximate weights of ropes in terms of their diameters are as follows:

Kind of Rope	Condition	Weight in Pounds per Foot, d in Inches
Manila	dry	$.27\,d^2$
Manila	wet or tarred . . .	$.32\,d^2$
Hemp	dry	$.29\,d^2$
Hemp	wet or tarred . . .	$.34\,d^2$
Cotton	dry	[illegible]

Fig. 181, where A denotes the axle and R, R one pair of antifriction wheels. If W denotes the load on the axle, the load transmitted to each antifriction wheel is $\frac{W}{2\cos\theta}$. Hence the friction on the axle of each antifriction wheel is $\frac{\mu W}{2\cos\theta}$, and on both is $\frac{\mu W}{\cos\theta}$. Let R denote the radius of the antifriction wheel, and r the radius of its axle. Then reducing the above value of the friction on the axle of the antifriction wheel to the surface of the wheel, that is, to the surface of the axle A, we have

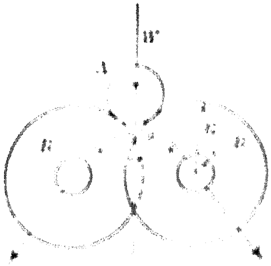


FIG. 181

$$\text{Frictional resistance on main axle} = \frac{\mu W r}{R\cos\theta}.$$

If the axle was running in an ordinary plain bearing, its frictional resistance would be μW. Hence the advantage of using a pair of antifriction wheels is expressed by the ratio:

$$e = \frac{\mu W}{\frac{\mu W r}{R\cos\theta}} = \frac{R\cos\theta}{r}.$$

The usual value of θ is $60°$, in which case

$$e = 0.87\frac{R}{r},$$

which is always greater than unity.

97. Roller and Ball Bearings. To reduce the frictional resistance of a plain bearing, rolling friction is now generally substituted for sliding friction in machine construction by the use of roller and ball bearings. In a rolling bearing, if the bearing is free from grit, there is no wear as in a sliding bearing, but destruction occurs by crushing of the surfaces in actual contact. From experiments on roller and ball bearings, made by Professor Streibeck of Germany, it has been found that the frictional resistance is least for balls rolling between perfectly flat surfaces, giving two points of contact. Increasing the number of points of con-

tact to three or four produced a higher frictional resistance without materially increasing the carrying capacity. Since failure occurs by crushing, it was found desirable, however, to curve the race, as this resulted in a large increase in carrying capacity with a very slight increase in frictional resistance. It is customary to make the transverse radius of the cups, or ball races, from $1\frac{1}{8}$ to $1\frac{1}{4}$ the radius of the balls.

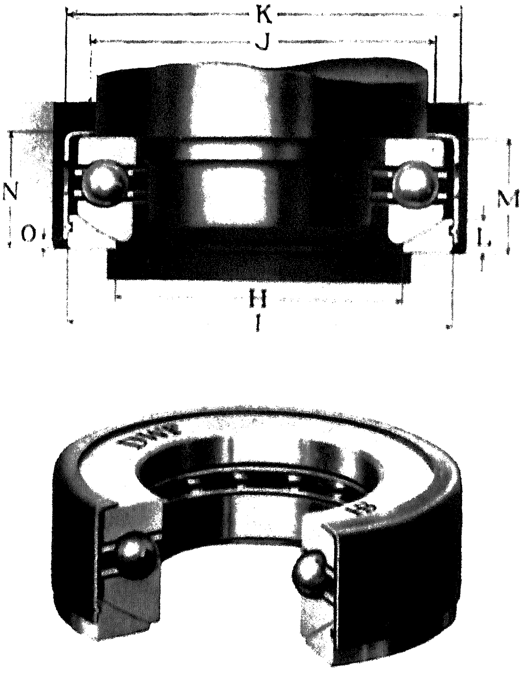


FIG. 182

The size of ball to be used in any given case is determined by the load and the crushing strength of the balls. The strength of the balls depends of course on the material and method of manufacture. It has been found by experiment that the crushing load L in pounds for one ball is given approximately by the formula

$$L = 82{,}400\ D^2,$$

where D denotes the diameter of the ball in inches. The safe load to be used in designing may be obtained by dividing the

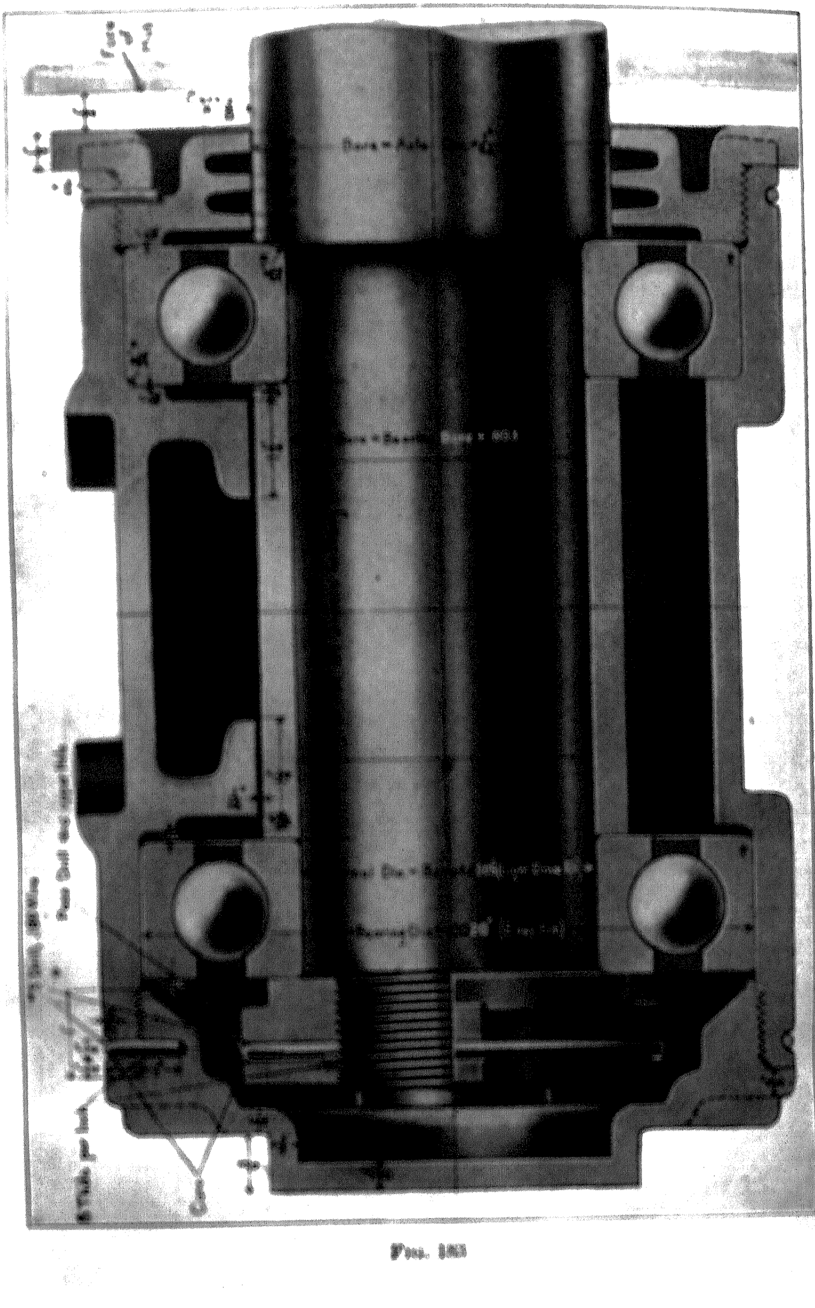


FIG. 185

crushing load L given by this formula by a factor of safety, varying from 120 for speeds not exceeding 50 r. p. m. to 775 for speeds of 2000 r. p. m.

In a thrust ball bearing, one type of which is shown in Fig. 182, the load on each ball is the total load divided by the number of balls. In a journal bearing the balls directly under the load are most heavily loaded, Professor Streibeck having found that about one fifth of the balls carry practically all the load. The heaviest load P on any one ball is thus given by the formula

$$P = \frac{5W}{N},$$

where W denotes the total load on the bearing and N the number of balls. Conversely, if P denotes the *safe* load for one ball, the safe working load W for the bearing is given by

$$W = \frac{PN}{5}.$$

Professor Spooner of London, England, has found that the actual working load on a ball bearing varies inversely as the square root of the product of speed and load. He also found that a practical rule for proportioning the load for different speeds is to decrease the load inversely as the cube of the speed.

In Fig. 183 a type of ball bearing is shown designed for the main bearings of passenger cars in railway service. The inner bearing here shown is so designed as to take up the end thrust. In general, the thrust capacity of bearings of this type is about one quarter of their radial capacity for light and medium loads, and one tenth of their radial capacity for heavy loads.

When bearings of this type are subjected to shocks, as in railway service, they should be designed from 2 to $2\frac{1}{2}$ times stronger than for uniform loads.

The following example illustrates the method of determining size of bearings required for eight-wheeled cars:

Weight of car without passengers	46,400 lb.
Weight of passengers, car crowded, 150 lb. × 80	12,000 lb.
Weight of car crowded	58,400 lb.

4 motors and gear cases at [illegible]	. . . . .	5400 lb.
8 wheels	at [illegible] lb.	[illegible] lb.
4 axles	at 350 lb.	1400 lb.
4 gears	at 200 lb.	800 lb.
Total weight not on main journals	. . . . .	10,6[illegible]0 lb.

Maximum weight on journals . . . . 17,700 lb.
Total weight per journal (approx.) . . . . 6,000 lb.
or 3000 lb. per bearing.

Maximum speed of a city car is about 25 m. p. h., hence use a factor of 2.25 giving rating of bearing as 3000 × 2.25 = 6750 lb.

The dimensions of the bearing to be used with this load depend somewhat on the type selected. For the separated ball type, the dimensions would be approximately as follow:

Bore 3 in.; outside diameter $7\frac{1}{2}$ in.; width $1\frac{3}{4}$ in.; diameter of balls $1\frac{3}{16}$ in.

When no retaining ring or cage is used, the balls must fill the race except for a clearance of about 0.001 in. between each pair.

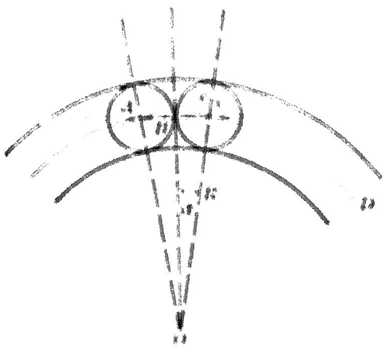


FIG. 184

To determine the size of the ball circle *ACD* (Fig. 184) consider two adjacent balls and connect their centers *A* and *C*. Then *B* on the line *AC* is their point of contact. Let *O* denote the center of the journal and α the angle *BOC*. Also let r denote the actual radius of the ball + the clearance (say 0.001 in.), and R the radius of the ball circle. Then

$$r = R \sin \alpha.$$

Also, since each ball subtends an angle 2α at the center, if N denotes the number of balls,

$$2\alpha N = 360°,$$

or

$$\alpha = \frac{180°}{N}.$$

Substituting this value of α in the equation for r and putting $\sin \alpha = \frac{1}{\operatorname{cosec} \alpha}$, the radius of the ball circle is found to be

$$R = r \operatorname{cosec} \frac{180°}{N}.$$

For heavy service a roller bearing is frequently used instead of a ball bearing, as it offers a considerably greater resistance to crushing for a bearing of the same size and weight. One type of roller bearing is shown in Fig. 185. The conical shape of this type of bearing not only permits of adjustment for wear, but also enables the bearing to take a certain amount of end thrust in addition to the radial load.

The coefficient of friction for roller bearings is very small and can be made less than 0.003 by proper design and lubrication. The safe load w per inch of total effective length of roller, assuming that one third of the rollers support the load, may be taken as

$$w\,(\text{lb./in.}) = 2000\ D^2,$$

where D denotes the mean diameter of the roller in inches. For a shaft of given diameter, the proper size of the rollers and the safe total load may be determined from the following empirical formulas:

$$D = 0.08\,D_s + \tfrac{3}{16}$$

for shaft diameters up to about 6 in., to the nearest sixteenth of an inch; and

$$W = 33{,}400\ \frac{D^2 N c}{S},$$

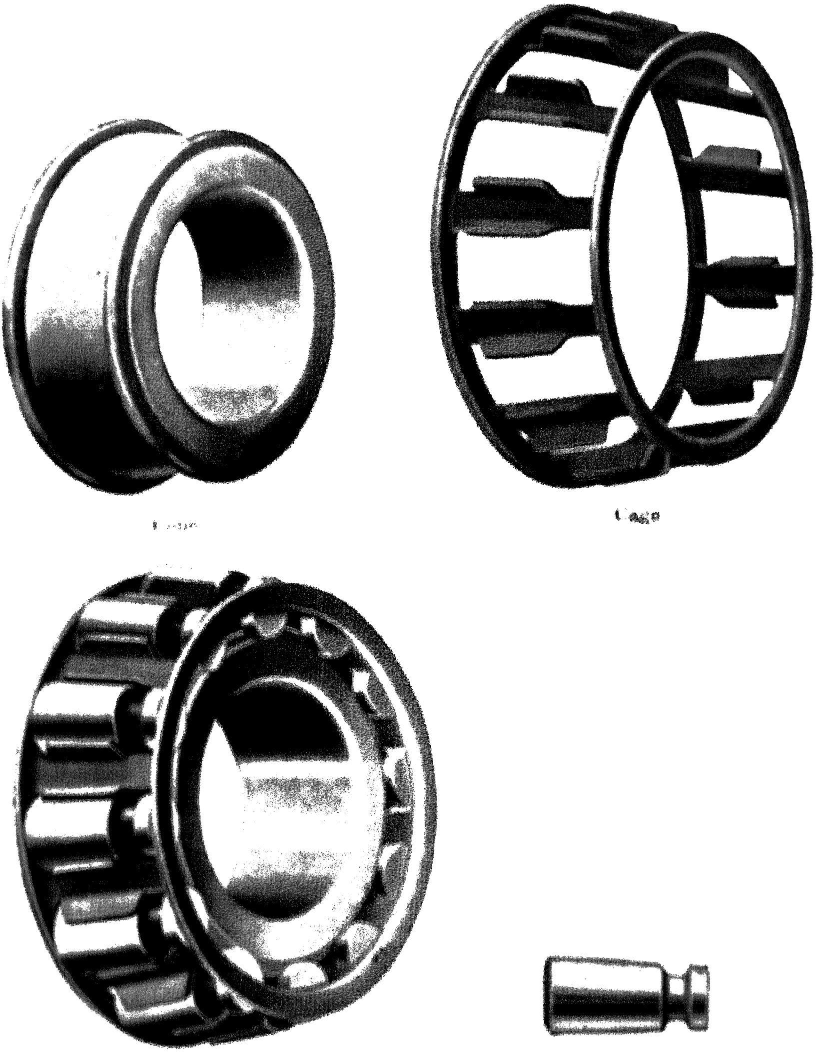
where D = mean diameter of roller in inches,
D_s = diameter of shaft, or bore of sleeve, in inches,
c = length of roller in inches,
N = number of rollers (not less than 6),
W = total safe load in pounds,
S = surface speed of convex-bearing surface or sleeve in ft./min.

PROBLEMS

295. A 3-in. shaft is carried by a roller bearing fitted with 20 hard steel half-inch rollers, each $4\frac{1}{2}$ in. long, and revolves at 140 r. p. m. What load can it carry with safety?

296. A ball bearing supports a total load of 1480 lb., and is fitted with two sets of 20 half-inch balls. Find approximately the maximum load on any one of the balls, and the factor of safety.

297. Twenty-four $\frac{3}{4}$-in. balls just fill a race, there being no cage and no clearance. Find the diameter of the circle of their centers.



Cone

Cage

Cone with Cage and Rolls Assembled

Roll

FIG. 185

98. Friction Transmission. — In friction transmission, spurs, miters, and bevels are commonly employed as with toothed gearing, the analogy between the two types of transmission in these three forms being quite general. In toothed gearing the teeth simply serve to insure positive relative motion, whereas in the

case of friction transmission the untoothed pitch surfaces of the mating wheels are employed, positiveness of relative motion being either undesirable or non-essential. The driving power of the wheels depends on the coefficient of friction between them, and hence their actual driving capacity becomes a function of the pressure with which they are held in contact.

It has been found that a paper driver mating with an iron follower forms one of the best combinations for durability and driving power. The paper friction is always used as the driver to prevent flat places being worn in either wheel in case of slippage. A pair of miter frictions is shown in Fig. 186. It has been found by experiment that for paper and iron frictions the most efficient working pressure is about 150 lb. per inch width of face, and that at this pressure a slippage of about 2 per cent occurs while driving loads corresponding to a coefficient of friction of 0.2. Using these experimental numerical values, the power transmitted is given by the formula

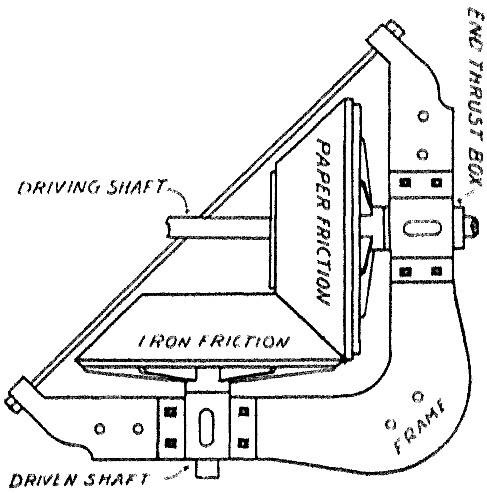


FIG. 186

$$\text{h. p.} = \frac{150\,w \times 0.2 \times \pi d n}{12 \times 33000},$$

which reduces to

$$\text{h. p.} = .000238\,wdn,$$

where w = width of face in inches (Fig. 187),

d = mean diameter in inches,

n = number of revolutions per minute.

One of the chief advantages of paper and iron frictions is that they can be used at much higher speeds than are practicable for

toothed gears. For toothed iron gearing a speed of 2200 ft./min. in the pitch line is excessive, and the same is true of 3000 ft./min.

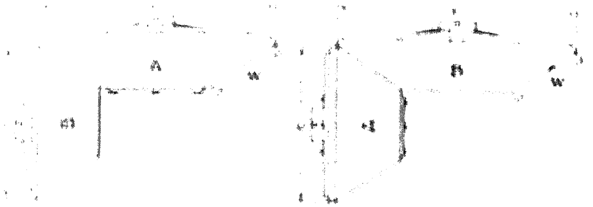


FIG. 187. Bevel Paper and Iron Friction Gearing.

for wood and iron toothed gears, whereas both are workable with friction transmission.*

For starting and stopping machinery quickly without jar or shock, and without changing alignment or position of shafting,

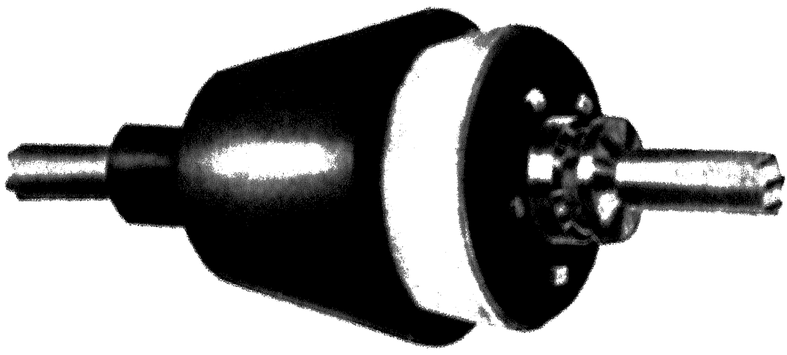


FIG. 188.

one of the simplest and most effective devices is the cone clutch, one form of which is shown in Fig. 188. The cone is faced either with paper or leather, the latter being used to a considerable ex-

* The following formula gives the horsepower transmitted by toothed gearing, based on a factor of safety of 8 and an ultimate tensile strength of 30,000 lb./in.²:

$$\text{h. p. for toothed gears} = \frac{wt^2v}{53\,l},$$

where
w = width of face in inches,
t = thickness of tooth at pitch line,
v = speed in ft./min. at pitch line,
l = length of tooth from point to root in inches.

tent in automobiles. For a paper cone the horsepower may be calculated by the formula given above. The half angle of the cone must not be less than the angle of repose for the materials in contact, for if less than this amount, the cone will wedge and require a reversed force to pull it out.* For a coefficient of friction $\mu = 0.15$ the half angle of the cone is $\theta = 8\frac{1}{2}°$ †; for $\mu = 0.20$, $\theta = 11\frac{1}{2}°$, and for $\mu = 0.25$, $\theta = 14°$. An average of about 10° is used in practice, although sometimes an angle as large as 14° is used.

The design of a cone clutch is based on the following analysis.

In Fig. 189 let P denote the normal pressure on the face of the clutch, μ the coefficient of friction, T the torque or moment which the clutch is transmitting, R the mean radius, and F the compression in the spring holding the clutch in place. Then

$$T = P\mu R,$$

and $$F = P(\mu \cos\theta + \sin\theta),$$

or, by eliminating P,

$$F = \frac{T}{\mu R}(\mu \cos\theta + \sin\theta).$$

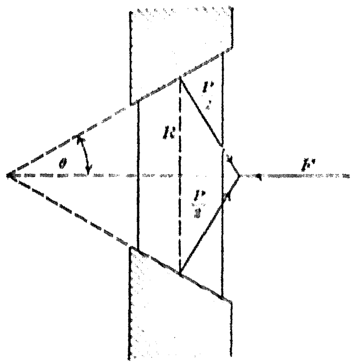


FIG. 189

For leather on cast iron assume $\mu = 0.25$, which corresponds to a half angle of $\theta = 15°$. For cast iron on cast iron assume $\mu = 0.175$, or $\theta = 10°$. If H denotes the number of horsepower being transmitted at a speed of N revolutions per minute, then

$$H = \frac{2\pi NT}{12 \times 33000}, \text{ or } T = \frac{63025\,H}{N}.$$

Hence from the first formula above, $T = P\mu R$, we have

$$P = \frac{T}{\mu R} = \frac{63025\,H}{N\mu R}.$$

Substituting $\mu = 0.25$ (leather on cast iron), this becomes

$$P = \frac{252100\,H}{NR}.$$

* Goss, *Trans. Am. Soc. Mech. Engrs.*, Vol. 18. † Note that $\mu = \tan\theta$.

Now if w denotes the width of the face in inches, p the allowable pressure in lb./in.², and $D = 2R$ = mean diameter of clutch, then

$$P = \pi Dwp,$$

or equating this to the value of P just above,

$$w = \frac{160427\,H}{ND^2p},$$

which gives the working relation between breadth and diameter.

To obtain an expression for the compression in the spring F, eliminate T between the equations

$$F = \frac{T}{\mu R}(\mu \cos\theta + \sin\theta),$$

and

$$T = \frac{63025\,H}{N},$$

in which case

$$F = \frac{63025\,H}{N\mu R}(\mu \cos\theta + \sin\theta).$$

Inserting $R = \frac{D}{2}$ and the numerical values $\mu = 0.25$, $\cos\theta = 0.9702$, $\sin\theta = 0.2425$, this becomes

$$F = \frac{244536\,H}{ND}.$$

One of the most efficient forms of clutch is that known as the multiple disk clutch, a simple form of which is shown in perspective in Fig. 190 and in cross section in Fig. 191. The sliding collar A is thrown in against the levers which exert pressure against the outside driving plate B. This plate moves in against the driven plate which is in turn moved in against the inside driving plate C. This is moved in against the second driven plate, etc. In the

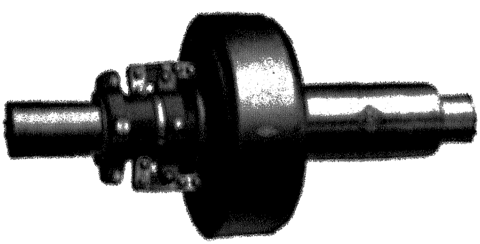


FIG. 190

clutch shown in Fig. 190 there are four frictional surfaces in contact when the clutch is in use. In automobile construction the number of friction rings, or disks, used is very large, as many as

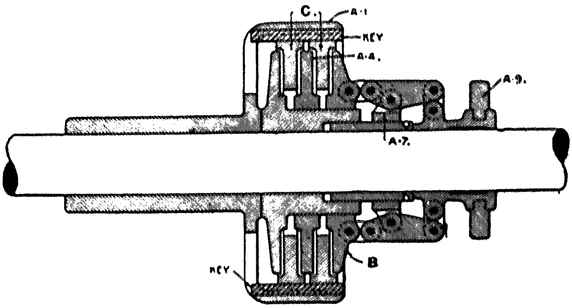


FIG. 191. — Multiple Friction Disks

70 friction surfaces being used in the best clutches. The object of using such a large number of friction surfaces is to decrease the intensity of the pressure, thus obtaining a clutch that will not "seize," but take hold gradually.

PROBLEMS

298. A motor cone clutch (leather on cast iron) transmits 20 h. p. at 900 r. p. m., the mean diameter of the clutch being 16 in. What should be the strength of the spring, and what is a suitable breadth for the leather surface, allowing a working pressure of 8 lb./in.²?

SOLUTION.
$$F = \frac{241536 \times 20}{900 \times 16} = 340 \text{ lb.}$$

$$w = \frac{160427 \times 20}{900 \times 16 \times 16 \times 8} = 1.74 \text{ in., say } 1\tfrac{3}{4} \text{ in.}$$

299. A motor running at 1000 r. p. m. transmits 10 h. p. through a cone clutch whose leather is 2 in. in breadth, the working strength of the spring being 110 lb. What should be the mean diameter of the clutch if the working pressure is not to exceed 7 lb./in.²?

300. In the paper and iron friction transmission shown in Fig. 187, calculate the horsepower transmitted for the dimensions and speeds shown in the following table.

	[illegible]	Large Diam. Inches	Small Diam. Inches	Speed R.P.M.	H.P.
I	2	6	2.67	2000	
II		4	1.78	3000	
I	3	9	4.01	1600	
II		6	2.67	2400	
I	4	12	5.34	[illegible]	
II		8	3.56	[illegible]	
I	5	[illegible]	6.68	800	
II		10	4.45	1200	
I	6	18	8.02	800	
II		12	5.35	1200	
I	7	18	[illegible]	[illegible]	
II		12	4.24	[illegible]	
I	8	24	[illegible]	[illegible]	
II		[illegible]	[illegible]	[illegible]	
I	[illegible]	[illegible]	[illegible]	[illegible]	
II		12	[illegible]	[illegible]	

99. Lubrication. It is of the utmost practical importance in machine design that proper provision be made for lubrication. The action of a lubricant is not fully understood, but it is supposed that the particles of the lubricant serve practically the same purpose as a ball bearing.

The object of lubrication in any case is to secure a film of the lubricant between the bearing surfaces. The kind of lubricant to be used depends on the class of service, mode of lubrication, and temperature of the bearing. The only general rule is that the heavier the bearing the heavier the lubricant that may be used.

It has been found by experiment that the wear on a bearing takes place where the film of oil is thinnest, and that this occurs at or near the point of greatest pressure. It has also been found that if an oil hole is drilled at the point of greatest pressure, the oil will be forced out of the hole, thus draining the bearing instead of feeding it. Hence, for ordinary lubrication the oil should be supplied to the bearing at a point where there is little or no pressure.

The most efficient form of oil lubrication is the forced feed, whereby a certain fixed amount of the oil is automatically supplied at regular intervals under pressure. In this way it is possible to maintain a constant film of oil of any desired thickness between the bearing surfaces. Forced lubrication may be said to be essential to such exacting conditions and high speeds as found in the footstep bearings of vertical steam turbines, and similar motors. With forced circulation of oil a pressure of one ton per square inch on the pivot of a 5-in. shaft has been found workable.*

100. Area of Bearings. — The safe working pressure on plain bearings depends largely upon the temperature at which they are maintained and the viscosity of the lubricant used. If the temperature of the bearing is raised sufficiently, the viscosity of the lubricant is so decreased that it is squeezed out. When this takes place and the metal surfaces come in contact, they will in general appear to weld themselves together, small pieces being torn out of both surfaces. This adhesion of the bearing surfaces is called **seizing**. The load at which seizing occurs depends upon the material, finish, and temperature of the bearing surfaces, but chiefly upon the viscosity of the lubricant. If this can be maintained by cooling the bearings, they will stand an enormous pressure without injury. Goodman cites a bearing with pad lubrication that was kept running for weeks under a load of 2 T./in.^{2} at a surface speed of 230 ft./min., the temperature being kept down to 110° F. by circulating water through the axle.

The first item to be observed in designing bearings is precision and stiffness of the loaded parts. Abnormal wear is frequently due to a misfit, or to springing of parts, whereby the pressure is localized and becomes very intense over a small

* For detailed information on lubrication the following references may be consulted:

Beauchamp Tower, *Proc. Inst. M. E.* (Great Britain), 1885.

Osborne Reynolds, *Phil. Trans.*

John Goodman, *Mechanics Applied to Engr.*, Chapter 7.

Hele-Shaw, *Cantor Lectures on Friction.*

W. F. M. Goss, "A Study in Graphite," Joseph Dixon Crucible Co.

area. For example the springing of a shaft will cause it to rest on the edges of the bearings, and produce rapid wear. This may be obviated by using spherical seated, or ball-and-socket bearings, such as are now commonly used for high speed motors and turbines.

The material to be used for a bearing depends upon the material of the shaft journal, the intensity of pressure, the speed and the lubricant. Cast iron may be used for moderate pressures, due chiefly to its porosity and absorptive power which causes the lubricant to adhere to it. Wrought iron and mild steel may also be used, while hard steel, if well ground, lubricated, and cooled, will carry an enormous pressure. More frequently, however, bearings are **Babbitted**, or lined with soft, white alloys. The advantage of using a soft material is that it flows under sufficient pressure, and thus distributes the pressure when it tends to become localized. For this reason seizing very rarely occurs in Babbitted bearings. The material generally used for ordinary machinery is bronze, or gun metal, an alloy of copper and tin in the proportion of 90 parts of copper to 10 of tin. The hardness of the alloy increases with the quantity of tin, so that for very great pressures the proportion of tin may be increased to 14 per cent or even 18 per cent. Conversely to produce a soft, tough bronze, suitable for the teeth of wheels subjected to shocks, the proportion of tin is decreased to 8 per cent. The hardness and resistance to wear is also increased by alloying with about 2 per cent of phosphorus, producing an alloy known as phosphor bronze. Some kinds of wood also make good bearings if provided with efficient water lubrication, *lignum vitæ* being commonly used for stern tube bearings of propellor shafts in vessels. According to Thurston, lignum vitæ and snakewood will sustain pressures exceeding 1000 lb./in.^{2}, where brass wears rapidly at one fourth this load; and will run continuously under 4000 lb./in.^{2}, at which load bronze seizes instantly. Camwood has also been found to work without injury under 8000 lb./in.^{2}

As mentioned above, the safe working pressure for bearings depends largely upon their temperature, as this conditions the effectiveness of the lubrication. The work done by friction is

largely converted into heat. Thus let

$W =$ total load on bearing,
$\mu =$ coefficient of friction,
$S =$ surface speed in ft./min. $= \dfrac{\pi d N}{12}$,
$N =$ number of revolutions per minute,
$d =$ diameter of bearing in inches,
$A =$ nominal or projected area of bearing surface in square inches.

Then the work done per minute in ft.-lb. is μWS, and the heat generated per minute in British thermal units is

$$\text{Heat generated} = \frac{\mu WS}{778} = \frac{\mu W \pi d N}{12 \times 778} \text{ B. t. u./min.}$$

Hence if r denotes the rate of cooling of the bearing, or number of heat units conducted away per minute, the required nominal area of the bearing becomes

$$A = \frac{\mu W \pi d N}{12 \times 778\, r}.$$

According to Goodman, the following may be assumed as average values of μ and r.*

Method of Lubrication	Coefficient of Friction μ	Conditions of Running	Rate of Cooling r	
			Pins	Continuous Running Bearing
Bath	0.004	Exposed to cool air (car axles, etc.)	4 to 7	1 to 1.5
Pad	0.012	Marine and stationary engines	0.75 to 1	0.3 to 0.5
Siphon	0.020	In hot places	0.4 to 0.5	0.1 to 0.3

According to Thurston, the liability of a journal to heating is not affected by changing the diameter, but is diminished by in-

* Goodman, *Mechanics Applied to Engineering*, p. 220.

creasing the length. He thus finds that the proper length of a bearing may be determined by the formula

$$l = \frac{WN}{\beta},$$

where β is an empirical constant determined by experiment, and that the safe working pressure p in lb./in.² is given by

$$p = \frac{\beta}{Nl}.$$

The following table of values of β has been computed by Unwin from the experiments of Beauchamp Tower: *

	Max. Pressure p in Bath	Empirical Constant, β
Sperm oil	415	717,000
Mineral oil	625	875,000
Rape oil	573	916,000
Olive and lard oil	520	936,000
Mineral oil	625	1,125,000
Rape oil fed by siphon	258	309,000
Rape oil fed with pad	328	393,000
Stationary engine crank pins	500	200,000
U. S. Navy		350,000
French Navy		100,000
Locomotives		1,000,000

After arriving at the safe bearing pressure by one of the methods given above, the result should be compared with the values given in the following table, which are based on actual practice.† The result may also be checked by Thurston's rule, which states that *the product of the surface speed in ft./min. and the pressure in lb./in.² should never exceed 50,000*.

* Unwin, *Elements of Machine Design*, Part I, p. 241. See also Benjamin, *Machine Design*, p. 105.

† Spooner, *Machine Design*, p. 96.

Kind of Bearing		Working Pressure Lb./In.²
Crank pins	small land engines	150–200
	marine engines	400–500
	fast land engines	500–800
	slow land engines	800–900
	torpedo boats	850–1000
	locomotives	1200–1800
	shearing machines	3000
	gas engines	350–400
Gudgeon pins, gas engines		800–1000
Crosshead neck journals		800–2100
Locomotive axle boxes	passenger	190
	freight	200
	tender	300–380
Main crank shaft bearings	freight steamers	200–225
	passenger steamers	225–300
	ironclads	250–350
	small cruisers	350–400
	steam tugs, etc.	400–550
Flywheel shaft bearings		150–250
Eccentric sheaves	stationary engines	60
	marine engines	70–140
Line shafting	gun metal steps	200
	cast-iron steps	50
Eccentric straps		70–140
Pivots	wrought-iron shaft on gun metal step	200–700
	cast-iron shaft on gun metal step	200–450
	wrought-iron shaft on lignum vitæ	1000–1400
Collar-thrust bearings for propeller shafts		50–80
Slides	cast iron on Babbitt metal	200–300
	cast iron on cast iron	40–100
Steel or iron shaft on lignum vitæ		350
Faces of link blocks		220–350
Pins of link blocks		550–1000

CHAPTER V

KINETICS OF PARTICLES

101. D'Alembert's Principle. In Chapter II, Art. 38, it was shown that Newton's laws of motion are embodied in the equation

$$F = m\frac{d^2s}{dt^2}.$$

This expression is therefore the fundamental equation of dynamics. Stated in words, it means that if a particle of mass m is acted upon by an external force F, an acceleration $\frac{d^2s}{dt^2}$ will be produced, of such an amount as to satisfy the above equality.

Now suppose that another external force P is applied to the particle, defined by the relation

$$P = -m\frac{d^2s}{dt^2},$$

where the acceleration $\frac{d^2s}{dt^2}$ is the same in amount and direction as above. Adding these two equations, we have

$$F + P = 0;$$

that is to say, the particle is in equilibrium under the external forces F and P. The force P so introduced is called the **kinetic reaction**, and depends upon the inertia of the particle, or its resistance to change of motion. The relation $F + P = 0$, or $F = -P$, is then simply the statement of Newton's third law, namely, that the kinetic relation is equal and opposite to the impressed force. From the relation $F = -P$, the kinetic reaction is often called the **reversed effective force**.

By introducing the idea of kinetic reaction, a dynamical problem is reduced to one in statics. That is to say, if in any dynamical problem the kinetic reactions are considered as well as

the external impressed forces, the problem can be solved by applying the ordinary conditions for static equilibrium. This important result is called **d'Alembert's Principle**, and is usually expressed in the form

$$F - m\frac{d^2s}{dt^2} = 0.$$

A more general expression of d'Alembert's Principle is obtained by resolving the impressed force F into components X, Y, Z, parallel to a system of rectangular axes, and also resolving the acceleration $\frac{d^2s}{dt^2}$ into components $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, parallel to the same axes. Then since the above relation holds for each pair of components separately, we have the three relations

$$X - m\frac{d^2x}{dt^2} = 0, \quad Y - m\frac{d^2y}{dt^2} = 0, \quad Z - m\frac{d^2z}{dt^2} = 0.$$

Now let δs denote any small displacement of the particle, and δx, δy, δz the rectangular components of δs. Then multiplying the three relations above by δx, δy, δz, respectively, and adding, they may be combined into one, namely,

$$\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z = 0.$$

For a system of n particles, this expression may be applied to each particle separately and their sum taken, in which case the general statement of d'Alembert's Principle becomes

$$\sum_1^n{}_r\left[\left(X_r - m_r\frac{d^2x_r}{dt^2}\right)\delta x_r + \left(Y_r - m_r\frac{d^2y_r}{dt^2}\right)\delta y_r + \left(Z_r - m_r\frac{d^2z_r}{dt^2}\right)\delta z_r\right] = 0.$$

In this general form the principle will be applied in the next chapter; as, for example, in deducing the law of the conservation of energy.

The practical importance of the principle, as mentioned above, consists in the fact that by introducing the kinetic reactions a dynamical problem may be solved by simply applying the conditions of static equilibrium, as illustrated by the examples in this and the following articles.

PROBLEMS

301. A man weighing 150 lb. stands in an elevator which starts to descend with an acceleration of 4 ft./sec.^{2} Determine the pressure of his feet on the floor of the elevator.

SOLUTION. The forces acting in this case are the weight of the man, 150 lb., acting vertically downward, his kinetic reaction, $m\frac{d^2s}{dt^2} = \frac{150}{g} \times 4$, acting vertically upward, and the reaction of the elevator, acting vertically upward. Calling the last reaction x, the condition of equilibrium becomes

$$x + \frac{150}{g} \times 4 - 150 = 0, \text{ whence } x = 131\tfrac{1}{4} \text{ lb.}$$

302. A weight of 5 lb. rests on a table and is connected by a string to a weight of 7 lb. which hangs over the edge of the table, as shown in Fig. 192. If the friction between the 5 lb. weight and the table amounts to ½ lb. and the friction of the pulley is neglected, determine the motion and the tension in the string.

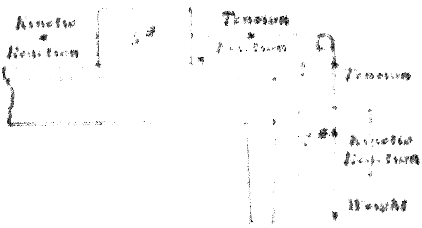


FIG. 192

SOLUTION. The impressed forces and kinetic reactions due to inertia are in this case as indicated in the figure. Considering first the system as a whole, the tension in the string does not appear since it is the same in both directions. Let a denote the acceleration. Then the total kinetic reaction for both weights is $\frac{5+7}{g}a$, the impressed force is 7 lb., and the frictional resistance is ½ lb. Hence applying the condition of equilibrium, $7 - \frac{1}{2} - \frac{5+7}{g}a = 0$, whence $a = 17\tfrac{1}{3}$ ft./sec.^{2}

To find the tension in the string, consider either weight separately. For the hanging weight the kinetic reaction is $\frac{7}{g} \times 17\tfrac{1}{3}$. Hence if T denotes the tension in the string, $T + \frac{7}{32} \times 17\tfrac{1}{3} - 7 = 0$, whence $T = 3.2$ lb. Similarly for the sliding weight we have $T - \frac{1}{2} - \frac{5}{32} \times 17\tfrac{1}{3} = 0$, whence $T = 3.2$ lb.

303. A train weighing 200 T. is accelerated from rest to 60 mi./hr. in 10 min. If the frictional resistance is 8 lb./T, find the draw-bar pull of the locomotive.

304. A locomotive exerts a constant draw-bar pull of 5 T. on a train weighing 200 T. up a 1 per cent grade. If the total frictional resistances amount to 10 lb./T., how long will it take to attain a velocity of 30 mi./hr. from rest, and how far will the train have moved?

305. The reciprocating parts of a locomotive weigh 500 lb. and attain a speed of 10 ft./sec. from either end of the stroke in 0.1 second. Find the average force exerted by the steam on the piston.

306. A rope passes over a smooth pulley. A man weighing 160 lb. hangs at one end of the rope and a weight of 180 lb. at the other. Find the uniform acceleration with which the man must pull himself up the rope in order to just balance the weight at the other end.

307. A man who can only lift 175 lb. can lift a barrel of flour weighing 196 lb. from the floor of an elevator when it is descending with a certain acceleration. What is the acceleration?

308. Two weights W_1 and W_2 are connected by a string passing over a smooth pulley. (Such an arrangement is called an Atwood's machine.) The whole apparatus is then placed on an elevator which starts to descend with an acceleration a. Find the tension in the string.

309. In the preceding problem find the tension in the string when the elevator starts to ascend with the same acceleration.

310. In an Atwood's machine one end of the string carries a weight W_1 and the other end carries a pulley of weight W_2 over which passes another string, having weights W_3 and W_4 at its ends. Find the tensions in both strings (Fig. 193).

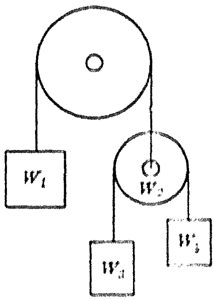


FIG. 193

NOTE. — The second pulley in this problem has a motion similar to that in the preceding problem.

102. Centrifugal and Centripetal Force. — When a body moves in a curved path, its motion at any instant is the same as if it was moving in a circle tangent to the path at the point in question, and of radius equal to the radius of curvature of the path at this point. If v denotes the speed at any instant, the actual motion can therefore be represented by a uniform circular motion of speed v.

It was shown in Art. 10, however, that for uniform circular motion, the body is accelerated toward the center, the amount of the acceleration being $a = \frac{v^2}{r}$, where r denotes the radius of the circle, or in case the path is not circular, its radius of curvature at the point in question. By Newton's law, the external force necessary to produce this acceleration is

$$F = m\frac{v^2}{r} = \frac{Wv^2}{gr}.$$

This is called the **centripetal force**, since it is directed toward the center. By d'Alembert's principle the kinetic reaction P which

equilibrates this force is

$$P = m\frac{v^2}{r} = \frac{Wv^2}{gr},$$

and is called the **centrifugal force** since it acts in the opposite direction to the impressed force; that is, away from the center.

For example, if a stone is fastened at the end of a string and then swung in a circle, the pull of the string on the stone is called the centripetal force, and the equal and opposite pull of the string on the hand, the centrifugal force. Again, if a train is rounding a curve, the pressure of the outer rail against the flanges of the wheels is the centripetal force, while the equal and opposite pressure of the wheel flanges against the outer rail is the centrifugal force. In the case of planetary motion around the sun, the attraction exerted by the sun on a planet is the centripetal force, and the attraction exerted by the planet on the sun is the centrifugal force.

PROBLEMS

311. A wheel revolving at 180 r. p. m. has a weight of 5 lb. fastened to it at 3 ft. from its axis and another weight of 7 lb. at 90° from the first and 2 ft. from the axis. Find the resultant centrifugal force on the axis. Also where must a 12-lb. weight be attached to the wheel to balance this centrifugal force?

312. A solid disk cast-iron flywheel, 2 in. thick and 3 ft. in diameter is 0.1 in. out of center. Find the outward pull exerted on the shaft when revolving at 200 r. p. m. Also where and how large a hole must be drilled in the wheel to balance it.

313. In a centrifugal railway, or "loop the loop," the car is constrained to move in a vertical circle by the inner pressure of a circular track. Find what the speed must be on entering the loop and at the highest point in order that the car shall make a complete circuit without leaving the rails.

SOLUTION. In order for the car to remain on the track the centrifugal force when at the highest point must not be less than its weight, that is, if v denotes the speed at the highest point,

$$\frac{Wv^2}{gr} \geqq W, \text{ whence } v \geqq \sqrt{gr},$$

where r denotes the radius of the loop. Since the speed V at the lowest point is given by the equation $V^2 = v^2 + 2g(2r)$,
the least value of V for which the car will make a complete circuit is obtained by substituting $v \geqq \sqrt{gr}$ in this relation, whence $V^2 \geqq gr + 4gr$, or

$$V \geqq \sqrt{5gr}.$$

Hence in a centrifugal railway, in order to loop the loop the speed on entering the loop, neglecting friction, must be greater than that obtained by running down an incline of height equal to $5r$, or $2\frac{1}{2}$ times the diameter of the loop

314. At what angle should a circular automobile speedway one mile in circumference be banked for a speed of 60 mi./hr. in order that there shall be no tendency to skid?

315. Find the velocity of projection in order that a bullet shot horizontally may travel round the earth continuously.

316. The center of a crank pin weighing 15 lb. is 8 in. from the center of the shaft. Find the centrifugal force developed at a speed of 150 r. p. m.

103. Application to Engine Governors, Railway Curves, etc. — If a body is swung in a horizontal circle at the end of a rod or string, one end of which is fastened at a point vertically above the center of the circle, as shown in Fig. 194, the arrangement is called a **conical pendulum.** A practical example of a conical pendulum is the ordinary Watt governor used on steam engines.

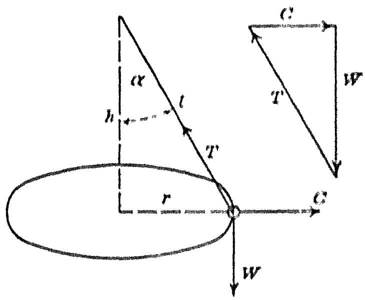


FIG. 194

Let W denote the weight of the body,* l the length of the suspension, T the tension in it, r the radius of the horizontal circle, v the speed with which the circle is described, C the centrifugal force developed at this speed, h the vertical height of the pendulum, and α the half angle of the cone (Fig. 194). Then

$$\tan\alpha = \frac{r}{h},$$

and from the force triangle

$$\tan\alpha = \frac{C}{W} = \frac{v^2}{gr}.$$

Hence $\frac{r}{h} = \frac{v^2}{gr}$, from which $h = \frac{gr^2}{v^2}$. Since $v = r\omega$, the expression

* The word "body" is used in this chapter to denote a single material particle, *i.e.* a finite mass concentrated at a single point.

for h in terms of the angular velocity ω becomes

$$h = \frac{g}{\omega^2}.$$

The height of a conical pendulum therefore depends only on its angular velocity, and is independent of its length or weight.

The period P, or time of making a complete revolution, is

$$P = \frac{\text{angle subtended by circumference}}{\text{angular velocity}} = \frac{2\pi}{\omega},$$

or, since from the above, $\omega = \sqrt{\frac{g}{h}}$, the period becomes

$$P = 2\pi\sqrt{\frac{h}{g}}.$$

The period is therefore the same as for a simple pendulum of length h. (See Art. 28.)

The practical application of the above to engine governors will be considered in detail in Chapter VII.

Instead of being suspended from a fixed point, a body may be constrained to move in a circular path by means of a pair of rails, as in the case of a train rounding a curve. In this case the resultant of the centrifugal force and the weight of the body is the reaction of the body on the ground or track. For the body to be in stable equilibrium, this resultant must fall within the supporting surface; that is to say, between the wheels or rails. Moreover, in the case of a railway train, if the resultant pressure is not normal to the plane of the track, it will have a lateral component which will cause friction between the outer rail and the flanges of the outer wheels, producing frictional resistance and wear. Hence to assure stability and avoid friction, the plane of the track is inclined at such an angle to the horizontal as to be perpendicular to the resultant of the weight and the centrifugal force at the ordinary speed at which trains pass around the curve. To a passenger in a car going around the curve at this speed, it will then appear to remain perfectly level, whereas at any other speed it will appear to tip.

To find the inclination of the track so that the reaction shall be normal to it at a given speed v, let r denote the radius of the

curve in feet and G the gauge of the track in inches (standard gauge $= 4$ ft. $8\frac{1}{2}$ in.). Then if α denotes the inclination of the track to the horizontal, from Fig. 195 (b),

$$\tan\alpha = \frac{C}{W} = \frac{v^2}{gr},$$

and from Fig. 195 (c),

$$\sin\alpha = \frac{H}{G},$$

where H denotes the difference in level of the rails in inches, or superelevation of the outer rail, as it is called.

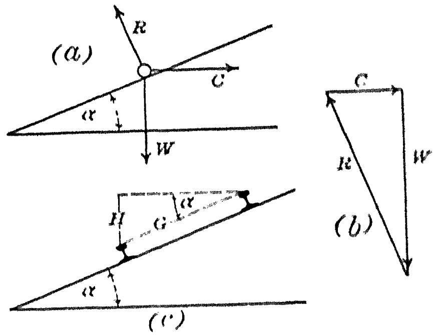


FIG. 195

Since α is always small, $\sin\alpha$ is approximately equal to $\tan\alpha$. Under this assumption

$$\frac{H}{G} = \frac{v^2}{gr},$$

whence

$$H = \frac{Gv^2}{gr}.$$

For standard gauge, namely $G = 4$ ft. $8\frac{1}{2}$ in., this becomes

$$H = 1.755\frac{v^2}{r}.$$

PROBLEMS

317. What must be the superelevation of the outer rail of a railway curve of 4 ft. $8\frac{1}{2}$ in. gauge for a speed of 50 mi./hr. on a curve of 800 ft. radius in order that there shall be no side thrust between the rails and wheel flanges?

318. The superelevation of the outer rail for a gauge of 4 ft. $8\frac{1}{2}$ in. is $2\frac{1}{2}$ in. for a speed of 45 mi./hr. What is the radius of the curve?

319. What is the proper speed for a curve of 900 ft. radius if the gauge is 4 ft. $8\frac{1}{2}$ in. and the superelevation of the outer rail is $1\frac{1}{4}$ in.?

320. The revolving ball of a conical pendulum weighs 10 lb. What is the speed for a height of 12 in.?

321. Find the change in height of a conical pendulum making 100 r. p. m. when the speed increases 1 per cent.

322. At what speed will a locomotive going round a level curve of 800 ft. radius exert a side thrust equal to 1 per cent of its own weight?

104. Harmonic Motion. — Consider a material particle which is subjected to a force directly proportional to its distance from a fixed point; that is, the impressed force is negative and directly proportional to the displacement. As shown in Art. 26, the motion is then a harmonic vibration, backward and forward through the point in question.

Let y denote the displacement measured from the fixed point as origin. Since by Newton's law the acceleration is proportional to the impressed force, we have

$$\frac{d^2y}{dt^2} \propto y.$$

If, then, the factor of proportionality is denoted by f, the differential equation of motion becomes

$$m\frac{d^2y}{dt^2} = -fy.$$

The constant factor f in this equation is the value of the impressed, or elastic force when the particle is at a unit's distance from the origin, *i.e.* when $y = 1$, and the minus sign is prefixed to indicate that the force always opposes the motion.

The general solution of a differential equation of this form is known to be

$$y = A \sin \omega t + B \cos \omega t,$$

where A and B are constants of integration, and ω is such as to satisfy the above differential equation. Thus, differentiating this expression twice, the second derivative is found to be

$$\frac{d^2y}{dt^2} = -\omega^2(A \sin \omega t + B \cos \omega t) = -\omega^2 y,$$

and comparing this result with the original differential equation, it is found that

$$\omega = \sqrt{\frac{f}{m}}.$$

To express the general solution in a more convenient form, let $A = r \cos \epsilon$ and $B = r \sin \epsilon$, where r and ϵ are two new constants whose values are to be determined. Making this substitution,

$$\begin{aligned} y &= r \cos \epsilon \sin \omega t + r \sin \epsilon \cos \omega t \\ &= r \cos (\omega t - \epsilon). \end{aligned}$$

Since the greatest value of the cosine is unity, the maximum value of y is r. This constant r is therefore called the **amplitude** of the motion. The other constant, ϵ, is called the **phase**.

If it is assumed that the displacement is zero when the time is zero, then substituting $y = 0$, $t = 0$ in the above equation as simultaneous values, the result is $r \cos(-\epsilon) = 0$, or since $r \neq 0$, $\cos(-\epsilon) = 0$, whence $\epsilon = \frac{\pi}{2}$. In this case,

$$\cos\left(\omega t - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \omega t\right) = \sin \omega t,$$

and the equation of motion becomes simply

$$y = r \sin \omega t.$$

Harmonic motion is therefore characterized by the fact that the displacement is a sine or cosine function of the time (compare Art. 26). Every periodic phenomenon in nature may be represented by a series of sine or cosine terms of this form, from the coördinates of a planet or satellite to the most complex vibrations of water, air, or ether, as represented by tides, rollers, ripples, breakers, sounds, radiations, luminous or otherwise, telegraphy with or without wires, earthquakes, temperature changes, etc.

Since the sine (or cosine) passes through the same series once in each revolution, that is, for an angular change of 2π, y will resume the same series of values after an interval of time in which the angle ωt has increased by 2π. Hence if P denotes the period, that is to say, the interval of time which elapses from the time when the particle is in any given position to that when it is again found in this position, then $P\omega = 2\pi$, or

$$P = \frac{2\pi}{\omega}.$$

Since $\omega = \sqrt{\frac{f}{m}}$, this may also be written

$$P = 2\pi\sqrt{\frac{m}{f}}.$$

Moreover from the equation $m\frac{d^2y}{dt^2} = -fy$ we have $\frac{m}{f} = -\frac{y}{a}$, where

a denotes the acceleration $\dfrac{d^2y}{dt^2}$. Hence the expression for the period may also be written

$$P = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

To find the velocity at any instant differentiate y with respect to the time. Then

$$v = \frac{dy}{dt} = r\omega \cos \omega t,$$

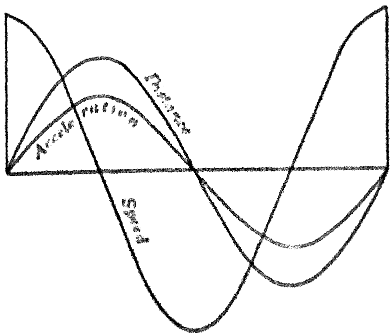


FIG. 196

or, since $\cos \omega t = \sqrt{1 - \sin^2 \omega t}$, this becomes

$$v = \omega\sqrt{r^2 - y^2}.$$

By plotting the displacement, velocity, and acceleration curves for harmonic motion to the same scale on a time base, as indicated in Fig. 196, the various characteristics of the motion are clearly shown.

PROBLEMS

323. The crank of an engine makes 250 r. p. m. and is $1\frac{1}{4}$ ft. long. Neglecting the obliquity of the connecting rod, that is, assuming the motion to be approximately harmonic, find the piston velocity and piston pressure when the crank has revolved through 45° from the "in end" of the stroke. Weight of reciprocating parts = 400 lb.

324. In the preceding problem, find the piston speed and piston pressure when the piston has moved forward 8 in. from the end of its stroke.

325. A weight rests on the scalepan of a spring balance which is given a vertical harmonic vibration of period 0.3 sec. What is the greatest possible amplitude of the motion in order that the weight may not leave the pan?

326. A pump plunger is operated by a crank whose pin works in a slotted sliding bar, as shown in Fig. 197. The plunger and attachments weigh 25 lb. and the crank revolves uniformly with a speed of 200 r. p. m. The total

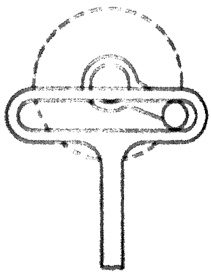


FIG. 197

stroke is 9 in., *i.e.* the radius of the crank circle is $4\frac{1}{2}$ in. Find the accelerating force on the plunger and its speed in ft./sec. when it is 2 in. from mid stroke.

327. In the preceding problem find the maximum accelerating force on the plunger and its maximum speed.

105. Elastic Vibrations. — The name "harmonic," as applied to elastic or natural vibrations, is due to their occurrence in producing musical sounds. Such vibrations are of common occurrence in nature because of the fact that in any system which is disturbed from a position of rest, forces are called into play which depend in general on the magnitude of the displacement. For instance, suppose that a certain displacement depends on a single variable, say y. In the case of elastic bodies, the force arising from this displacement will then be a uniform continuous function of y, say $F(y)$. Expanding this function by Maclaurin's theorem, we have

$$F(y) = F(0) + F'(0)y + \frac{F''(0)}{\underline{|2}}y^2 + \cdots$$

Assuming that the elastic force is zero when the displacement is zero (*i.e.* that the body was originally at rest) the first term in the development disappears, and consequently

$$F(y) = F'(0)y + \frac{F''(0)}{\underline{|2}}y^2 + \cdots$$

For a small displacement y, the terms in y^2, y^3, etc., may be neglected in comparison with that in y. Hence since $F'(0)$ is a constant, say C, the expression for the impressed force becomes simply

$$F = Cy$$

and the motion is therefore harmonic.

The simplest method of representing a harmonic motion is by compounding it with a uniform rectilinear motion. This is, in fact, the device used in self-registering instruments, such as the indicator of a steam engine, in which the paper is caused to move horizontally, while the tracing point has a vertical motion proportional to the force exerted. When this method is applied to a harmonic vibration, the result is a sinusoid, as shown in Fig. 128.

The equation of such a harmonic curve is evidently

$$y = r\cos(\omega t - mx),$$

where x and y are the horizontal and vertical coördinates of a point of the curve. When x is constant, the equation is identical with the integral in Art. 104, and represents a simple harmonic rise and fall with the time at the particular place in question. When t is constant, it gives an instantaneous glance at the whole wave, as shown in Fig. 198.

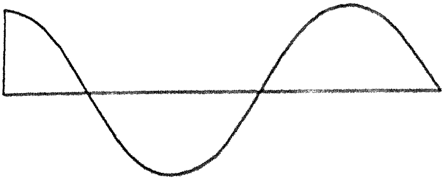


FIG. 198

The equation of an equal wave moving in the opposite direction is

$$y = r\cos(\omega t + mx).$$

If these two waves are superposed, as in the case of a vibrating string fixed at the ends, where each wave when it reaches the end of the string is reversed in direction and reflected back, the resulting wave has for its equation

$$y = r\cos(\omega t - mx) + r\cos(\omega t + mx).$$

Expanding and simplifying, this reduces to

$$y = 2r\cos\omega t\cos mx.$$

The points of the string for which $\cos mx = 0$ remain undisturbed by the vibration, and are called **nodes**. In stringed instruments such nodes may be created artificially by stopping the vibration of the string at a certain point by touching it lightly with the finger. The length of the wave is thereby changed and consequently the pitch is altered, the resulting tone being technically known as a **harmonic**.

106. Application to Beams, Rods and Springs. There are numerous practical instances of bodies having harmonic motion, for in perfectly elastic bodies the straining force is proportional to the displacement produced (Hooke's law), and most substances are almost perfectly elastic over a limited range.

For example, consider a light, stiff beam, and let h denote its deflection at the center under a load W at rest at this point (Fig. 199). For simplicity, the weight W is assumed to be large in comparison with the weight of the beam, so that the latter may be neglected. If the load W is suddenly applied to the beam, or dropped on it, the deflection will be greater than h, and since it cannot retain this deflection under a static load W, the beam will vibrate up and down until the energy of the impact is destroyed by molecular friction and other resistances. From Art. 104 the equation of motion is

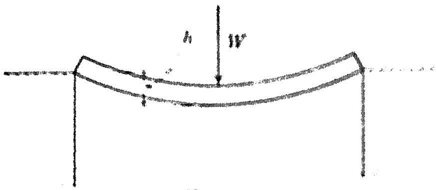


Fig. 199

$$m\frac{d^2y}{dt^2} = -fy,$$

where f denotes the value of the elastic force at a unit's distance from the position of no load, or unstrained position. If, then, the unit load is taken as 1 lb. and the unit distance as 1 ft., f is the load in pounds per foot of displacement, *i.e.*

$$f = \frac{W}{h}.$$

The period of vibration is then

$$P = 2\pi\sqrt{\frac{m}{f}},$$

or since $m = \frac{W}{g}$ and $f = \frac{W}{h}$, this becomes simply

$$P = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}},$$

where the acceleration in this case is that due to gravity (compare Art. 26). The beam therefore vibrates in the same time as a simple pendulum of length h. (See Art. 28.)

The same law applies to the longitudinal vibrations of a rod as well as to its transverse vibrations. For example, consider a vertical rod supported at its upper end and having a nut or head at

the lower end upon which a sliding weight falls, as shown in Fig. 200. If h denotes the deflection of the rod when the weight W is at rest, the impact of its fall will cause it to deflect an amount greater than h, and since it cannot retain this deflection, it will vibrate longitudinally. If the stress does not pass the elastic limit of the material, the motion will therefore be harmonic, and its period will be, as above,

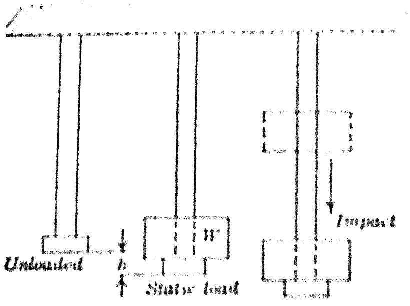


FIG. 200

$$P = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{\text{static displacement}}{\text{acceleration due to gravity}}}.$$

The same result applies to a spring of any form. For example, the graduations on a spring balance are at equal distances apart, which indicates that the force is proportional to the stretch or displacement. If, then, a helical spring stretches h in. under a load of W lb. and the spring is displaced from its position of equilibrium and then released, it will execute a vibration of period

$$P = 2\pi\sqrt{\frac{h}{g}}.$$

NOTE. — h and g must be expressed in the same units, either both inches or both feet.

A flat spring, such as shown in Fig. 201, or an elliptical or semi-elliptical spring, such as is in common use on vehicles, is evidently simply a special case of the light stiff beam considered above. It is therefore only necessary to note the static deflection of the spring under a given load, such as the weight of an automobile or carriage body, in order to determine the period of its vibration when the load is suddenly applied, as for instance when a vehicle is in motion. It will therefore vibrate with the same period as a simple pendulum of length h, although the *amplitude* of the

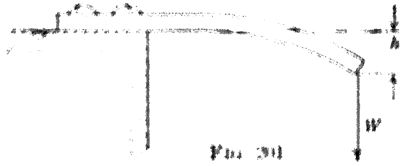


FIG. 201

vibration depends on the amount of the disturbance, as in the case of a simple pendulum.

Torsional vibrations are also of the same nature as the above, but as their chief mechanical application is in finding moments of inertia, their consideration will be reserved for the next chapter, where this subject is discussed (Art. 126).

PROBLEMS

328. A light spiral spring elongates ½ in. under a weight of 2½ lb., 1 in. under a weight of 5 lb., etc. How many complete oscillations per minute will it make with a 6-lb. weight attached?

329. A semi-elliptic automobile spring deflects 1½ in. under a load of 1 T at its center. Find the period of vibration of the spring when so loaded.

330. The helical spring of an automobile shock absorber deflects 1 in. under a load of 500 lb. How many vibrations per minute will it make when loaded with 1500 lb.?

331. A vertical tie rod of a bridge deflects ¼ in. under a wheel load of 10 T. If the train passes over the bridge at high speed, find the period of vibration of this rod.

107. Composition of Collinear Harmonic Motions. Consider the composition of two harmonic vibrations in the same line, given by the equations

$$y_1 = r_1 \cos(\omega t + \epsilon_1), \quad y_2 = r_2 \cos(\omega t + \epsilon_2),$$

where the amplitudes and phases are different, but the periods are the same; namely, $P = \dfrac{2\pi}{\omega}$. The equation of the resultant motion is then

$$y = r_1 \cos(\omega t + \epsilon_1) + r_2 \cos(\omega t + \epsilon_2).$$

Expanding and simplifying this expression, it becomes

$$y = (r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2) \cos \omega t - (r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2) \sin \omega t.$$

To simplify this still further, introduce two new constants R and E, defined by the relations

$$r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2 = R \cos E,$$

$$r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2 = R \sin E.$$

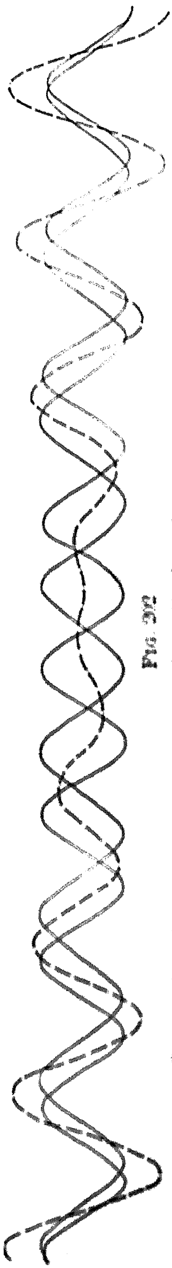


FIG. 202

Composition of two collinear waves of equal amplitude and wave lengths in the ratio of 8 to 9.

Then the expression for y becomes

$$y = R \cos E \cos \omega t - R \sin E \sin \omega t = R \cos(\omega t + E).$$

The resultant motion is therefore also a harmonic vibration, of amplitude R and phase E.

The values of R and E may be found from the equations defining them. Thus by squaring and adding, since $\sin^2 E + \cos^2 E = 1$, we have

$$R = \sqrt{(r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2)^2 + (r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 + 2 r_1 r_2 \cos(\epsilon_1 - \epsilon_2)}.$$

Again, by division of the expressions defining R and E, R drops out and the left member becomes $\tan E$,

whence $$E = \tan^{-1} \frac{r_1 \sin \epsilon_1 + r_2 \sin \epsilon_2}{r_1 \cos \epsilon_1 + r_2 \cos \epsilon_2}.$$

By comparing the equation for y with those for y_1 and y_2, the period of the resultant vibration is obviously equal to that of its components; namely, $P = \dfrac{2\pi}{\omega}$.

108. Waves and Tides. — One of the best examples of the composition of collinear harmonic motions is that furnished by the ocean tides. Here the sun and moon if acting separately would each produce a tide, or harmonic rise and fall of the ocean, once in about 24 hours. The tides actually occurring are then the resultant of these two effects. Thus when the sun and moon are in conjunction, as at new moon, or in opposition, as at full moon, the whole tide is the sum of the solar and lunar tides, and we have what are called spring tides. When the moon is in quadrature, it is low tide as regards the sun, whereas it is high tide as regards the moon, the actual tide being equal to the excess of the lunar over the solar tide. In the first

and third quarters high tide is earlier than if due to the moon alone, and in the second and fourth quarters it is later, this phenomenon being known as the priming and lagging of the tides.

A similar phenomenon is sometimes observed on an ocean beach when two sets of waves of slightly different period run together, the result being that every ninth or tenth wave is of unusual height. This is shown graphically in Fig. 202, the full lines representing the component waves, and the broken line their resultant.

109. Composition of Harmonic Motions at Right Angles. — Consider two harmonic motions of the same period P $\left(\text{or frequency } \frac{1}{P}\right)$ in directions at right angles. The equations of motion are then

$$x = r_1 \cos(\omega t + \epsilon_1), \quad y = r_2 \cos(\omega t + \epsilon_2).$$

To obtain the equation of the path of the resultant motion of which these are the components, eliminate t between these equations. From the first equation

$$\frac{x}{r_1} = \cos(\omega t + \epsilon_1), \text{ whence } \omega t = \cos^{-1}\frac{x}{r_1} - \epsilon_1.$$

Inserting this value of ωt in the second equation, the result is

$$\frac{y}{r_2} = \cos\left[\cos^{-1}\frac{x}{r_1} - \epsilon_1 + \epsilon_2\right]$$

$$= \cos\cos^{-1}\frac{x}{r_1}\cos(\epsilon_2 - \epsilon_1) - \sin\cos^{-1}\frac{x}{r_1}\sin(\epsilon_2 - \epsilon_1).$$

Let $\cos^{-1}\frac{x}{r_1}$ be denoted by θ. Then, as indicated in Fig. 203,

$$\sin\cos^{-1}\frac{x}{r_1} = \sin\theta = \frac{\sqrt{r_1^2 - x^2}}{r_1},$$

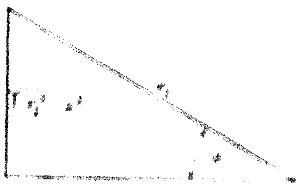


Fig. 203

and consequently the above expression becomes

$$\frac{y}{r_2} = \frac{x}{r_1}\cos(\epsilon_2 - \epsilon_1) - \frac{\sqrt{r_1^2 - x^2}}{r_1}\sin(\epsilon_2 - \epsilon_1).$$

Transposing and squaring in order to get rid of the radical, this simplifies into

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} - \frac{2\,xy}{r_1 r_2}\cos(\epsilon_2 - \epsilon_1) = \sin^2(\epsilon_2 - \epsilon_1),$$

which represents an ellipse with semiaxes r_1 and r_2, as shown in Fig. 204. The resultant motion in this case is called **elliptic harmonic.**

If the difference in phase, $\epsilon_2 - \epsilon_1$, is zero, the equation becomes

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} - \frac{2\,xy}{r_1 r_2} = 0,$$

or

$$\left(\frac{x}{r_1} - \frac{y}{r_2}\right)^2 = 0,$$

which represents a pair of coincident straight lines; namely, one of the diagonals of the rectangle circumscribing the ellipse shown in Fig. 204. The motion in this case is therefore rectilinear and simple harmonic.

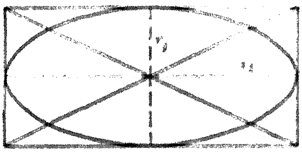


FIG. 204

Similarly, if the difference in phase is 180°, that is, $\epsilon_2 - \epsilon_1 = \pi$, the equation of the path breaks up into the pair of coincident straight lines

$$\left(\frac{x}{r_1} + \frac{y}{r_2}\right)^2 = 0,$$

which represents the opposite diagonal to the preceding.

If the amplitudes are equal and the phase difference is 90°, that is, if $r_1 = r_2$ and $\epsilon_2 - \epsilon_1 = \frac{\pi}{2}$, then the path becomes the circle

$$x^2 + y^2 = r^2,$$

and we have uniform circular motion.

When the periods of the component motions are nearly but not quite equal, the phase of one motion gains gradually on the other, and the path passes continuously through the forms of all possible ellipses inscribed in the rectangle constructed on $2\,r_1$ and $2\,r_2$ as sides. The path in this case is a kind of spiral, which touches in succession each side of the rectangle or square.

When the periods of the component motions are unequal, the equation of the path is not, in general, algebraic. Only when the ratio of the periods (or frequencies) is a rational number will the path be reentrant and algebraic.

To illustrate the latter, consider the simple case when one period is twice the other, and let the phase difference be denoted by ϵ. Then the equations of the component motions are

$$x = r_1 \cos(\omega t + \epsilon), \quad y = r_2 \cos 2\,\omega t.$$

Since $\cos 2\,\omega t = 1 - 2\sin^2 \omega t$, the latter equation may be written

$$y = r_2(1 - 2\sin^2 \omega t), \text{ or } y = r_2(2\cos^2 \omega t - 1),$$

whence $\sin^2 \omega t = \dfrac{r_2 - y}{2\,r_2}$, $\cos^2 \omega t = \dfrac{y + r_2}{2\,r_2}$.

Expanding the expression for x, and inserting the values of $\sin \omega t$ and $\cos \omega t$ just obtained, the result is

$$x = r_1(\cos \omega t \cos \epsilon - \sin \omega t \sin \epsilon)$$

$$= r_1\left(\sqrt{\frac{y + r_2}{2\,r_2}} \cos \epsilon - \sqrt{\frac{r_2 - y}{r_2}} \sin \epsilon\right),$$

which reduces to

$$\frac{x^2}{r_1^2} = \frac{y}{2\,r_2} \cos 2\,\epsilon + \tfrac{1}{2} - \tfrac{1}{2} \sin 2\,\epsilon \sqrt{1 - \frac{y^2}{r_2^2}}.$$

Rationalizing this expression, it becomes an equation of the fourth order, representing a curve called a lemniscate which resembles a figure 8 as shown on the left in Fig. 205. If $\epsilon = \dfrac{\pi}{2}$, this curve becomes a parabola, as shown on the right. One of the intermediate positions is also shown in the figure.

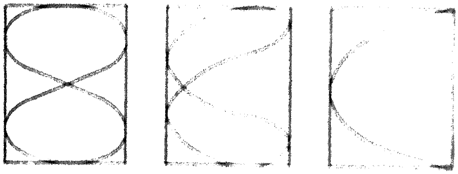


FIG. 205

The curves shown above are known as Lissajous' curves, and have important applications in acoustics. They may be obtained mechanically by means of a piece of apparatus known as a Black-

burn's pendulum. This consists simply of three threads, arranged as shown in Fig. 206, two of the threads being fastened to supports at A and B, and joined at C to the third, which carries a tracing point, or dropper, at its lower end D. For a small disturbance in the plane of the paper, the point D will evidently move like the bob of a simple pendulum of length CD, that is, it will execute an approximately harmonic vibration of period $2\pi\sqrt{\frac{CD}{g}}$; whereas for a disturbance perpendicular to the plane of the paper, the period

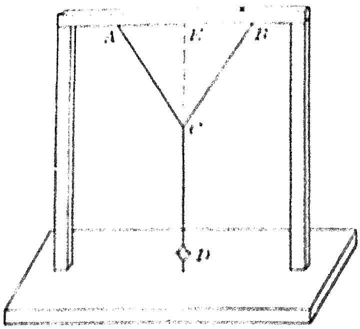


FIG. 206

of the vibration will be $2\pi\sqrt{\frac{ED}{g}}$. (See Art. 28.) The amplitude of the motion and the difference of phase evidently depend on how the bob D is set in motion. This simple apparatus therefore offers a means of producing all possible combinations of harmonic motions at right angles. To eliminate the resistance of the air as far as possible the threads should be long so as to make the motion very slow. A convenient way to accomplish this is to attach the threads to the ceiling and let the bob swing near the floor. The bob should be of lead, and the tracing point may consist of a funnel with small opening, filled with sand, or a tube filled with ink and with a fine orifice at the lower end. By laying a sheet of paper under the bob, the curves will then be traced on it as the pendulum swings.

110. Damped Vibrations. — In discussing elastic vibrations the body has so far been considered as perfectly free to move, in which case it will execute a harmonic vibration, as explained in Art. 105. Perfect freedom of motion, however, is an ideal assumption which is never fully realized in practice, owing to the resistance of the air and other causes. In the case of an actual solid, vibration is resisted by internal molecular friction as well as by the inertia or viscosity of the air or other medium through which it moves. To this resistance is due the fact that actual vibrations

do not continue indefinitely, but die away after a longer or shorter interval of time.

It has been found by experiment that the external resistance varies directly as the speed for small oscillations, and the internal resistance may also be assumed to follow the same law with a sufficient degree of approximation. Assuming, then, that the damping resistance is proportional to the velocity $\frac{dy}{dt}$, it becomes $k\frac{dy}{dt}$ where k is a constant of proportionality, and the differential equation of motion given in Art. 104 for simple harmonic motion now becomes

$$m\frac{d^2y}{dt^2} = -fy - k\frac{dy}{dt}.$$

The damping resistance $k\frac{dy}{dt}$ is negative, since, like the elastic force fy, it always resists motion. The solution of this differential equation is known to be of the form

$$y = Ae^{\alpha t} + Be^{\beta t},$$

where A and B are constants of integration, and α and β are to be determined by the condition that the solution shall satisfy the given differential equation. Making this substitution, we have

$$m(\alpha^2Ae^{\alpha t} + \beta^2Be^{\beta t}) + f(Ae^{\alpha t} + Be^{\beta t}) + k(A\alpha e^{\alpha t} + B\beta e^{\beta t}) = 0,$$

or, separating the coefficients of A and B,

$$Ae^{\alpha t}(m\alpha^2 + f + k\alpha) + Be^{\beta t}(m\beta^2 + f + k\beta) = 0.$$

Since this relation must be satisfied for arbitrary values of A and B, the coefficients of A and B must each be zero; that is,

$$m\alpha^2 + f + k\alpha = 0,$$
$$m\beta^2 + f + k\beta = 0.$$

Since these equations are identical in form, α and β are solutions of the quadratic

$$mz^2 + f + kz = 0.$$

Solving this equation, the roots are

$$z = -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m^2} - \frac{f}{m}}.$$

Let this radical be denoted by c; that is, let $c = \sqrt{\frac{k^2}{4m^2} - \frac{f}{m}}$. Then the values of α and β become

$$\alpha = -\frac{k}{2m} + c, \quad \beta = -\frac{k}{2m} - c,$$

and hence the general solution is

$$y = Ae^{-\frac{k}{2m}t}e^{ct} + Be^{-\frac{k}{2m}t}e^{-ct}.$$

Two cases must now be distinguished, according as the radical c is real or imaginary.

CASE I. Radical real, *i.e.* $\frac{k^2}{4m^2} - \frac{f}{m} > 0$, or $k > 2\sqrt{mf}$. Since y is not a periodic function of the time, the motion in this case is **non-periodic.** It is therefore not a vibration.

Let v_0 denote the initial velocity of the point; that is, let $\frac{dy}{dt} = v_0$ for $t = 0$. Then

$$v_0 = A\left(-\frac{k}{2m} + c\right) + B\left(-\frac{k}{2m} - c\right).$$

Also from the condition that $y = 0$ for $t = 0$, we have

$$A + B = 0.$$

Solving these two equations simultaneously for A and B, the result is

$$A = \frac{v_0}{2c}, \quad B = -\frac{v_0}{2c},$$

and inserting these values in the general solution it becomes

$$y = \frac{v_0}{2c}e^{-\frac{k}{2m}t}(e^{ct} - e^{-ct}).$$

As t increases, e^{ct} increases, always remaining greater than unity, and e^{-ct} decreases, always remaining a proper fraction. Hence the expression in parenthesis never alters its sign. Moreover, for $t = \infty$, $y = 0$; that is, the motion dies away as time goes on. Such motion occurs when the damping is very strong; as, for example, in the case of a ballistic pendulum, or in the ordinary dash pot

used to check the motion of valves, etc. The general characteristics of such motion are shown in Fig. 207.

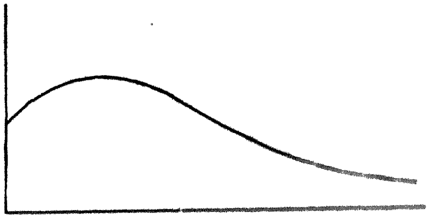


FIG. 207

CASE II. Radical imaginary, *i.e.* $k < 2\sqrt{mf}$. In this case let $c' = \sqrt{\frac{f}{m} - \frac{k^2}{4m^2}}$. Then $c' = ic$, where i denotes $\sqrt{-1}$, and the general solution becomes

$$y = \frac{v_0}{c'} e^{-\frac{k}{2m}t} \frac{e^{ic't} - e^{-ic't}}{2i}$$

$$= \frac{v_0}{c'} e^{-\frac{k}{2m}t} \sin c't.$$

This evidently differs from an undamped harmonic vibration $y = A \sin \omega t$ by the occurrence of the exponential factor $e^{-\frac{k}{2m}t}$. When t is small, this exponential factor differs but little from unity, and consequently there is but small change in the amplitude. As t increases, however, the change in the amplitude increases rapidly. The general character of the motion is shown in Fig. 208.

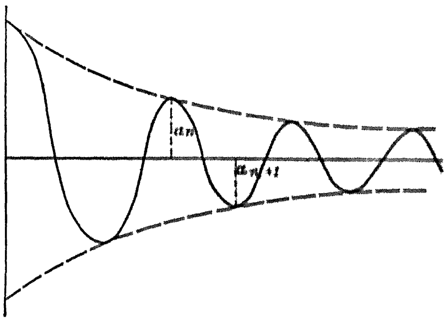


FIG. 208

For the period of the vibration, we have

$$P = \frac{2\pi}{c'} = \frac{4\pi m}{\sqrt{4mf - k^2}},$$

which is independent of the time and the amplitude. The speed, therefore, decreases, since it takes equal times to pass over portions of the path which are constantly decreasing in length.

From this expression for P it is also evident that the period for a damped vibration is greater than for one which is undamped; namely, when $k = 0$.

The speed is obtained from $v = \dfrac{dy}{dt}$, which in the present case gives

$$v = \frac{dy}{dt} = \frac{v_0}{c'} e^{-\frac{k}{2m}t}\left(c' \cos c't - \frac{k}{2m} \sin c't\right).$$

At the extreme displacement $v = 0$; hence

$$c' \cos c't - \frac{k}{2m} \sin c't = 0, \text{ or}$$

$$\tan c't = \frac{2mc'}{k} = \frac{\sqrt{4mf - k^2}}{k}.$$

The same value of the tangent is obtained when the angle $c't$ is increased by any multiple of π. Hence the interval of time between successive maxima and minima is always the same and equal to $\dfrac{\pi}{c'}$; that is to say, half the period P.

From $\tan c't$ the sine may be calculated by means of the trigonometric formula $\sin x = \dfrac{\tan x}{\sqrt{1 + \tan^2 x}}$. Thus in the present case

$$\sin c't = \sqrt{\frac{4mf - k^2}{4mf}}.$$

Inserting this value in the expression for y, we obtain the maximum excursion, or what corresponds to the amplitude in the case of a simple harmonic vibration. Let this be denoted by a_n, where n refers to the nth excursion since starting. Then

$$a_n = \frac{v_0}{c'} e^{-\frac{kt}{2m}} \sqrt{\frac{4mf - k^2}{4mf}}.$$

Similarly, the amplitude of the next excursion, a_{n+1}, is given by

$$a_{n+1} = \frac{v_0}{c'} e^{-\frac{kt'}{2m}} \sqrt{\frac{4mf - k^2}{4mf}},$$

where $t' = t + \dfrac{P}{2} = t + \dfrac{2\pi m}{\sqrt{4mf - k^2}}$. Hence

$$a_{n+1} = a_n e^{-\frac{kP}{4m}} = a_n e^{\frac{-\pi k}{\sqrt{4mf - k^2}}}.$$

Since the ratio $\frac{a_{n+1}}{a_n}$ is constant, successive excursions form a geometric series. Taking natural logarithms, we have

$$\log a_n - \log a_{n+1} = \frac{\pi k}{\sqrt{4mf - k^2}} = \frac{kP}{4m}.$$

The logarithms of successive excursions therefore always differ by a constant quantity. This constant difference is called the **logarithmic decrement.**

By observing successive amplitudes a_n and a_{n+1} and the period P, the damping factor k may be calculated from this formula. Or, if successive amplitudes on the same side are observed, the logarithmic decrement will be twice as great as here given; namely, $\frac{kP}{2m}$. This affords a simple means for calculating the damping resistance in any actual case.

PROBLEM

332. A body weighing one ton rests on a spring which executes a damped vibration. It is found by experiment that the elastic resistance amounts to 35 lb. for $\frac{1}{2}$ in. deflection, and that the damping resistance is 50 lb. at a speed of 5 ft./sec. Calculate the logarithmic decrement, and determine whether or not the motion is periodic. Also determine what the damping resistance must be in order that the motion shall constitute the limit between periodic and non-periodic.

Solution. Here

$$f = \frac{35 \text{ lb.}}{\frac{1}{24} \text{ ft.}} = 840 \text{ lb./ft.}, \quad k = \frac{50 \text{ lb.}}{5 \text{ ft./sec.}} = 10 \frac{\text{lb. sec.}}{\text{ft.}}$$

and

$$m = \frac{2000 \text{ lb.}}{32.2 \text{ ft./sec.}^2} = \frac{1000}{16.1} \frac{\text{lb. sec}^2}{\text{ft.}}.$$

Consequently $c = \sqrt{\frac{k^2}{4m^2} - \frac{f}{m}} = \sqrt{0.00648 - 13.524}$, which is imaginary. The motion is therefore periodic. Also logarithmic decrement $= \frac{\pi k}{\sqrt{4mf - k^2}} =$ 0.06879. In order for the motion to be the limiting case between periodic and non-periodic, the damping factor k must equal $2\sqrt{mf}$. Hence in the present case its value would be

$$k = 2\sqrt{mf} = 456.8 \frac{\text{lb. sec.}}{\text{ft.}}$$

111. Forced Vibrations. — The theory of forced vibrations is of great importance in the explanation of many physical phenomena. For example, if two tuning forks of the same pitch are placed near

together and one is set in vibration, the other will vibrate also. This phenomenon is called **resonance.** The same phenomenon is observed in electric oscillations, and is the principle on which wireless telegraphy is based. The rolling of a ship is another instance of forced oscillation. If, when resting in still water, a ship should be displaced from its position of equilibrium, it would, of course, oscillate with a certain period which may be called its natural period of oscillation. When the ship is acted on by waves, if the impulses have the same period as the natural period of the ship, its oscillations may become so great as to be dangerous. The same is true of the oscillations produced by a column of soldiers marching in step across a bridge. In this case, if the period of their step happens to be the same, or nearly the same, as the natural period of vibration of one of the bridge members, this member may be forced into such violent vibrations as to be wrecked.

To obtain a solution of the problem in question, assume that in addition to the elastic resistance and the damping resistance, the body is also acted upon by a periodic force, say,

$$F \sin \eta t,$$

where F denotes the greatest value of this exciting force and η depends on its period of oscillation. Under these conditions, the differential equation of motion becomes

$$m\frac{d^2y}{dt^2} + fy + k\frac{dy}{dt} = F \sin \eta t,$$

the left member being identical with the equation of the last article. The general solution of an equation of this type is obtained by finding a particular solution, and adding to this the solution of the above equation when the right member is zero. The latter, however, is identical with the solution obtained in the preceding article for damped oscillations; namely,

$$y = Ae^{-\frac{k}{2m}t}e^{\sigma t} + Be^{-\frac{k}{2m}t}e^{-\sigma t}.$$

To obtain a particular solution, assume one of the form

$$y = D \sin (\eta t + \phi).$$

where D and ϕ are constants to be determined from the condition that this solution shall satisfy the given differential equation. The first two derivatives of this expression are

$$\frac{dy}{dt} = D\eta \cos(\eta t + \phi), \quad \frac{d^2y}{dt^2} = -D\eta^2 \sin(\eta t + \phi),$$

and inserting these values in the given differential equation, the condition that they shall satisfy it is found to be

$$-mD\eta^2 \sin(\eta t + \phi) + fD \sin(\eta t + \phi) + kD\eta \cos(\eta t + \phi) = F \sin \eta t.$$

Expanding the trigonometric functions, this becomes

$$\sin \eta t \left[-mD\eta^2 \cos\phi + fD\cos\phi - kD\eta \sin\phi - F\right]$$
$$+ \cos \eta t \left[-mD\eta^2 \sin\phi + fD\sin\phi + kD\eta\cos\phi\right] = 0.$$

Since this relation must hold for all values of t, it must be independent of t; that is to say, each quantity in brackets must be zero. From the first, we have

$$D = \frac{F}{\cos\phi(f - m\eta^2) - k\eta\sin\phi};$$

and from the second,

$$\tan\phi = \frac{k\eta}{m\eta^2 - f}.$$

If, then, these values of D and ϕ are inserted in the assumed particular solution, namely,

$$y = D\sin(\eta t + \phi),$$

this equation will represent at least a possible form of forced vibration, which will be its actual form if the initial conditions are properly fulfilled. The two other terms entering into the general equation represent the solution of the problem when $F = 0$; that is, in the case of the natural, unforced vibrations of the body. The particular solution just obtained, which represents the effect of the external exciting force, is therefore of special importance.

The particular solution given above represents a simple undamped oscillation, whose period is independent of the mass of

the body, its elastic resistance, or its natural period of vibration. Since $\sin(\eta t + \phi)$ differs from $\sin \eta t$, the greatest excursion does not occur at the same time that the exciting force assumes its greatest value, but lags behind it. The two functions, $\sin \eta t$ and $\sin(\eta t + \phi)$, however, assume the same *series* of values, and like values always differ by the same interval of time; namely, $t = \frac{\phi}{\eta}$; *i.e.*, there is a difference in phase between F and the forced oscillation. Since $\tan\phi = \frac{k\eta}{m\eta^2 - f}$, the phase difference ϕ is small if the damping k is small. If there is no damping, $k = 0$ and consequently $\phi = 0$.

If the denominator $m\eta^2 - f = 0$, that is, $\eta = \sqrt{\frac{f}{m}}$, then $\tan\phi = \infty$ and $\phi = \frac{\pi}{2}$. Hence in this case the forced oscillation has the same period as the natural undamped period of the body. In this case there is said to be **complete resonance** between the two. If we also have $k = 0$, then $\tan\phi = \frac{0}{0}$, but since the damping is never entirely lacking, this indetermination never actually occurs.

The case when η is equal to, or nearly equal to, ω (Art. 104) is of special importance. The constant D represents the amplitude of the forced vibration, and since

$$D = \frac{F}{\cos\phi(f - m\eta^2) - k\eta\sin\phi},$$

this amplitude is proportional to the exciting force F. If $f - m\eta^2$ and k are both infinitesimal, the denominator becomes indeterminate. To evaluate this indetermination write

$$\cos\phi = \sin\phi\cot\phi = \frac{m\eta^2 - f}{k\eta}\sin\phi.$$

Then
$$D = -\frac{F}{\sin\phi\left[\frac{(f - m\eta^2)^2}{k\eta} + k\eta\right]}.$$

For the case of resonance, $\phi = \frac{\pi}{2}$ and $f - m\eta^2 = 0$. Hence in this case

$$D = -\frac{F}{k\eta} = -\frac{F}{k}\sqrt{\frac{m}{f}}$$

If the damping k is small, this quantity is large. This is the explanation of the well-known phenomenon that very great oscillations are produced whenever the period of the exciting force is equal to, or nearly equal to, the natural period of the body on which it acts, and the damping resistance acting on the latter is small.

The complete solution of the original differential equation with which we started is made up of the sum of the particular solution

$$y_1 = D\sin(\eta t + \phi)$$

and the complementary function

$$y_2 = Ae^{-\frac{k}{2m}t}e^{ct} + Be^{-\frac{k}{2m}t}e^{-ct}.$$

The latter solution was discussed in the preceding article, where in the present case c is imaginary. To express this solution in the trigonometric form, make use of the relations

$$e^{ix} = \cos x + i\sin x, \quad e^{-ix} = \cos x - i\sin x,$$

where i denotes $\sqrt{-1}$. Then since $c' = ic$ (Art. 110), this expression becomes

$$y_2 = e^{-\frac{k}{2m}t}\left[A(\cos c't + i\sin c't) + B(\cos c't - i\sin c't)\right]$$
$$= e^{-\frac{k}{2m}t}\left[(A + B)\cos c't + (A - B)\, i\sin c't\right].$$

In order for this expression to be real, the arbitrary constants A and B must be complex quantities. Therefore let $A + B = B_1$ and $(A - B)i = A_1$. Then

$$y_2 = e^{-\frac{k}{2m}t}(A_1 \sin c't + B_1 \cos c't),$$

and the complete solution becomes

$$y = e^{-\frac{k}{2m}t}\left[A_1 \sin c't + B_1 \cos c't\right] + D\sin(\eta t + \phi).$$

Since the first term contains the exponential factor $e^{-\frac{k}{2m}t}$, the absolute amount of this term decreases with the time, whereas the last term does not change so long as the exciting force continues to be exerted. The damping therefore causes the natural vibra-

tions of the body to die away until finally it vibrates in accordance with the form given by the particular integral y_2.

Now consider the case in which the body was originally in equilibrium, that is, when $t = 0$, and is gradually brought into oscillation by the action of some external harmonic vibration. In this case the constants A_1 and B_1 must be determined by the initial conditions, namely, that when $t = 0$, $y = 0$, and also the velocity $\frac{dy}{dt} = 0$. The first of these conditions gives

$$B_1 + D \sin \phi = 0.$$

Also, since

$$\frac{dy}{dt} = -e^{-\frac{k}{2m}t}\left\{\sin c't\left(\frac{k}{2m}A_1 + c'B_1\right) + \cos c't\left(\frac{k}{2m}B_1 - c'A_1\right)\right.$$
$$\left. + D\eta \cos (\eta t + \phi)\right\},$$

the second condition gives

$$-\frac{k}{2m}B_1 + c'A_1 + D\eta \cos \phi = 0.$$

Solving these two equations simultaneously for A_1 and B_1, the results are

$$A_1 = -D\left(\frac{\eta}{c'} \cos \phi + \frac{k}{2mc'} \sin \phi\right),$$

$$B_1 = -D \sin \phi.$$

At the beginning of the motion the damping resistance is small and may be neglected. Under this assumption, namely, $k = 0$, and inserting the values just found for A_1 and B_1 in the complete solution, it becomes

$$y = D\left\{-\frac{\eta}{c'} \cos \phi \sin c't - \sin \phi \cos c't + \sin (\eta t + \phi)\right\}.$$

Under the assumption that $k = 0$ we also have $\tan \phi = 0$, or $\phi = 0$ and $D = \frac{F}{f - m\eta^2}$. Inserting these values, the above equation simplifies into

$$y = \frac{F}{f - m\eta^2}\left\{-\frac{\eta}{c'} \sin c't + \sin \eta t\right\}.$$

Since $c' = \sqrt{\frac{f}{m} - \frac{k^2}{4m^2}}$ (Art. 110), when $k = 0$, we have $c' = \sqrt{\frac{f}{m}}$ $= \omega$ (Art. 104), and consequently the above may also be written

$$y = \frac{F}{m(\omega^2 - \eta^2)} \left\{ \sin \eta t - \frac{\eta}{\omega} \sin \omega t \right\}.$$

From the form of this equation it is evident that the motion consists of two simple harmonic oscillations, one having the same period as the natural period of the body, and the other that of the impressed oscillation. The resultant motion is therefore the algebraic sum of the two, this phenomenon being known as **interference.** (Compare Art. 108.)

In the case of resonance, ω differs but little from η. Consequently the expression before the brace in the last equation is very large. The expression within the brace, however, is very small at the beginning of the motion, but increases with the time. Hence as time goes on the oscillations are successively very small and very large. The condition for a maximum or minimum is $\frac{dy}{dt} = 0$, which gives in this case $\cos \eta t = \cos \omega t$. Hence $\sin \eta t = \pm \sin \omega t$. The positive sign here gives the small amplitudes, and the negative sign the large ones. In acoustics this successive increase and decrease in the amplitude of vibration, or strength of tone, are called **beats.**

If the damping k is not small, or if the time is so far advanced that the factor $\frac{kt}{2m}$ is noticeably different from zero, we must go back to the original equation. The beats are therefore only noticeable at the beginning of the motion. Later on, the natural vibrations die away, and there remains only those produced by the impressed force.

PROBLEM

333. In Prob. 332 suppose that the body is acted on periodically by an external impressed force of 1 lb. at intervals of 3 sec. How great will the amplitude of the forced oscillations be after so long a time that the natural vibrations of the body have died away? How does this compare with the amplitude of its natural vibration?

112. Plane Pendulum. — When a single material particle is suspended from a weightless fiber and allowed to swing, the arrangement is called a **simple plane pendulum.** It is evidently impossible to realize such an ideal piece of apparatus in practice. However, if a small heavy body like a lead bullet is hung at the end of a long fine thread, its motion for small oscillations will be approximately that of a simple pendulum, and ordinary pendulums approach the same motion more or less closely according to their construction. The motion of a simple pendulum is chiefly useful as a standard for comparison in the discussion of elastic vibrations, and for that reason is here considered.

It was shown in Art. 28 that the period of a simple pendulum is given approximately by the formula

$$P = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

The exact value of the period will now be obtained by applying the principle of work and energy.

Let α denote the half angle of swing, l the length of the pendulum, and θ the angle between the pendulum and the vertical at any particular instant, as shown in Fig. 209. Then in swinging from A to B, the work done on the weight W is Wh, where h denotes the difference in elevation of the two points. If, then, v denotes the speed at A, by equating the work done to the energy acquired, we have

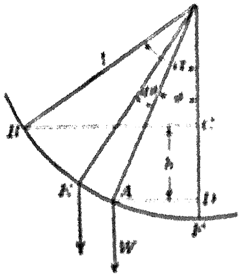


FIG. 209

$$\frac{Wv^2}{2g} = Wh, \text{ or } v = \sqrt{2gh}.$$

Since $h = l\cos\theta - l\cos\alpha$, this becomes

$$v = \sqrt{2gl(\cos\theta - \cos\alpha)}.$$

If, then, AE denotes the length of path described in an interval of time dt, we have $AE = vdt$. Also from the figure, $AE = ld\theta$.

Hence $vdt = ld\theta$, from which

$$dt = \frac{l}{v}\,d\theta = \frac{l\,d\theta}{\sqrt{2\,gl(\cos\theta - \cos\alpha)}} = \sqrt{\frac{l}{2\,g}}\,\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

To find the entire time of swing from F to B, integrate this expression between the limits 0 and α. The period, or time of making a complete oscillation, will then be four times this amount, whence

$$P = 4\sqrt{\frac{l}{2\,g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

To transform this integral into a more convenient form, make use of the trigonometric relation $\cos x = 1 - 2\sin^2\frac{x}{2}$. The expression for P then becomes

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} = 2\sqrt{\frac{l}{g}}\int_0^{\alpha}\frac{d\theta}{\sin\frac{\alpha}{2}\sqrt{1 - \left(\dfrac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}\right)^2}}.$$

Now introduce a new variable ϕ, defined by the relation

$$\sin\phi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}.$$

Differentiating this expression, $\cos\phi d\phi = \dfrac{\cos\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right)}{\sin\frac{\alpha}{2}}$, whence

$$d\theta = \frac{2\sin\frac{\alpha}{2}\cos\phi d\phi}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\alpha}{2}\cos\phi d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}}.$$

To determine the limits of integration for the new variable ϕ, we have $\theta = \alpha$ when $\phi = \frac{\pi}{2}$, and $\theta = 0$ when $\phi = 0$. Hence

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{2 \sin\frac{\alpha}{2}\cos\phi d\phi}{\sin\frac{\alpha}{2}\sqrt{1-\sin^2\phi}\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}},$$

or, finally,

$$P = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}}.$$

The value of this integral cannot be obtained directly, but may be determined to any desired degree of accuracy by expanding the quantity under the integral sign into a series and integrating this series term by term. The integral was called by Legendre an elliptic integral of the first kind, and will be found fully treated in any standard work on elliptic functions. Numerical values of the integral will also be found tabulated in such works, from which the following brief table is taken for use in applying the formula.

$\frac{\alpha}{2} =$	0°	2.5°	5°	10°	20°	30°	45°	60°	75°	90°
$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}} =$	1.5708	1.5715	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	∞

To illustrate the use of the table, suppose that a pendulum is 10 in. long, and $\alpha = 5°$. Then $\frac{\alpha}{2} = 2.5°$, and the value of the integral is found from the table to be 1.5715. Hence the period is

$$P = 4\sqrt{\frac{l}{g}}(1.5715) = 4\sqrt{\frac{\frac{10}{12}}{32.2}}(1.5715) = 1.01125 \text{ sec.}$$

PROBLEMS

334. Find the required length for a seconds pendulum if the arc of swing is 10°.

335. Solve the preceding problem by the approximate formula $P = 2\pi\sqrt{\frac{l}{g}}$, and compare the results.

113. Approximate Pendulum Formula. — From the result of the preceding article, a simple approximate formula may be obtained which is comparatively accurate for small angles of swing, say, $\alpha < 30°$, and which may be used when the simple approximation deduced in Art. 28 does not suffice.

Since the sine of a small angle is also small, the two being of the same order of magnitude, the quantity $\sin^2 \frac{\alpha}{2} \sin^2 \phi$ in the expression for P is very small, and decreases rapidly when raised to higher powers. Let this small quantity be denoted by k. Then the quantity under the integral sign in the elliptic integral for P may be written $\frac{1}{\sqrt{1-k}}$, or, expanding by the binomial theorem,

$$\frac{1}{\sqrt{1-k}} = (1-k)^{-\frac{1}{2}} = 1 + \frac{k}{2} + \frac{3k^2}{8} + \cdots.$$

Neglecting powers of k higher than the first, we have therefore

$$P = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\left(1 + \frac{k}{2}\right)d\phi$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} d\phi + 2\sqrt{\frac{l}{g}}\sin^2\frac{\alpha}{2}\int_0^{\frac{\pi}{2}}\sin^2\phi\, d\phi.$$

The value of the first integral in this expression is simply $\frac{\pi}{2}$. The second integral, $\int_0^{\frac{\pi}{2}} \sin^2 \phi\, d\phi$, is one of frequent occurrence, especially in the theory of electricity. Its value may be found most simply by considering the average value of the function. Thus, as ϕ passes from 0° to 90°, the sine passes through all values from 0 to 1. Likewise the cosine passes through the same series of values in the reverse order. Therefore since $\sin^2 \phi + \cos^2 \phi$ always equals unity, the average value of each function between 0 and $\frac{\pi}{2}$ is $\frac{1}{2}$. Hence

$$\int_0^{\frac{\pi}{2}} \sin^2 \phi\, d\phi = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{4}.$$

Substituting this value for the integral, the expression for the period now becomes

$$P = 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{\pi}{8}\sin^2\frac{\alpha}{2}\right).$$

If α is small, $\sin\frac{\alpha}{2}$ in this expression may be replaced by its arc $\frac{\alpha}{2}$, in which case it becomes

$$P = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\alpha^2}{16}\right).$$

This formula differs from the simple approximate formula $P = 2\pi\sqrt{\frac{l}{g}}$, derived in Art. 28, by the term $2\pi\sqrt{\frac{l}{g}}\frac{\alpha^2}{16}$. This quantity is therefore the correction which must be applied when accuracy is desired.

PROBLEM

336. Apply the above correction to the result in Prob. 334 and compare with the result in Prob. 335.

114. Cycloidal Pendulum. — For a simple pendulum it was shown in Art. 113 that the period depends on the amplitude of the motion. Only for small oscillations is the period approximately independent of the arc through which it swings.

If, however, the point moves in the arc of a cycloid instead of a circular arc, the time of swing is the same, no matter what the amplitude may be. For this reason the motion in this case is said to be **isochronos** and the cycloid is called the **tautochrone**.

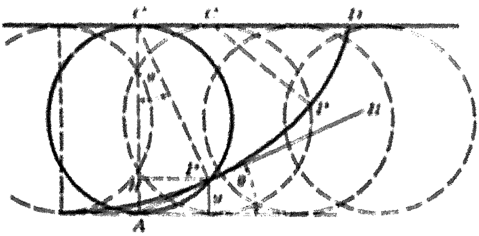


FIG. 210

A cycloid is the curve traced by a fixed point on the circumference of a circle rolling along a straight line, as shown in Fig. 210. Its characteristic difference from the circle is that in the latter the center of curvature is a fixed point, whereas in the cycloid it changes continuously. Thus, in Fig. 211, every point of the circular arc AB is equidis-

tant from the center C, and this fixed point C is the center of curvature for the entire curve. For the cycloidal arc AD, however, the center of curvature lies on the line CP through the instantaneous center C (Fig. 210), and changes continuously. The locus of these instantaneous centers C_1, C_2, C_3, etc. (Fig. 211), corresponding to the points of the curve P_1, P_2, P_3, etc., is called the **evolute,** from the fact that if a pattern, or template, is cut having the form of the curve CD, and a string is then laid along the edge of this pattern and fastened at C, the end D when the string is unrolled will describe the cycloidal arc DA.

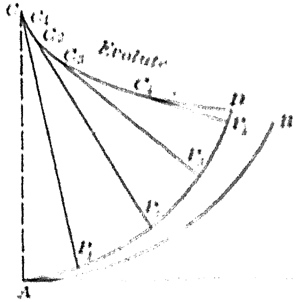


FIG. 211

This method of generating the curve is of practical importance because it gives a means of constructing a cycloidal pendulum. Thus, if two guides are constructed having the form of the evolute, or centrode, CD, and a heavy particle, such as a small lead ball, is attached to C by a thin fiber of length $CD = CA$, its motion will be that of the ideal cycloidal pendulum except for a slight damping effect due to the resistance of the air and the weight of the suspending fiber.

The differential equation of the cycloid is easily deduced from the way in which the curve is generated; that is, by rolling a circle along a straight line. Thus in Fig. 212, the instantaneous center for any point P of the curve is the point of contact C of the rolling circle and the fixed straight line. Since the angle APC is inscribed in a semicircle, it is a right angle. Hence AP is perpendicular to CP, and consequently AP is tangent to the cycloid at P. If, then, θ denotes the angle between the tangent AB and the horizontal, the slope of the curve at the point P is

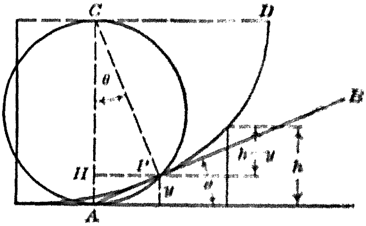


FIG. 212

$$\frac{dy}{dx} = \tan\theta.$$

Now
$$\tan\theta = \frac{PH}{CH},$$

or, since $PH^2 = CH \times HA = (2r - y)y$, and $CH = 2r - y$, where r denotes the radius of the rolling circle, this may be written,

$$\tan\theta = \frac{\sqrt{y(2r - y)}}{2r - y}.$$

Hence the differential equation of the cycloid is

$$\frac{dy}{dx} = \sqrt{\frac{y}{2r - y}}.$$

To find the period of oscillation, the principle of work will be applied, as in the case of the simple pendulum (Art. 112). Suppose that the particle starts from a point at the height h above its lowest position (Fig. 212). Then its velocity at any other height y will be

$$v = \sqrt{2g(h - y)}.$$

Since $v = \frac{ds}{dt}$, we have, by inserting this value of v,

$$dt = \frac{ds}{v} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2g(h - y)}} = \sqrt{\frac{1 + \left(\frac{dx}{dy}\right)^2}{2g(h - y)}}\,dy.$$

Substituting in this equation the value of $\frac{dx}{dy}$ from the differential equation of the curve, the expression for the time becomes

$$dt = \sqrt{\frac{r}{gy(h - y)}}\cdot dy.$$

Hence by integration the time of swing from the highest to the lowest point is

$$t = \sqrt{\frac{r}{g}}\int_0^h \frac{dy}{\sqrt{hy - y^2}} = \sqrt{\frac{r}{g}}\,\text{vers}^{-1}\frac{2y}{h}\Big|_0^h = \pi\sqrt{\frac{r}{g}}.$$

The period $P = 4\,t$, which may be written

$$P = 2\,\pi\sqrt{\frac{4\,r}{g}}.$$

Since the expression for t is the exact value of the above integral and is independent of the height h, the period is the same no matter what the value of h. That is to say, the time of swing is independent of the point on the cycloid from which the bob falls, and consequently the motion is strictly isochronos.

It is also noteworthy that the formula for P is the same as that obtained for a simple pendulum of length $l = 4\,r$. The reason for this is that the radius of curvature CA of the cycloid at the vertex A is four times the radius of the generating circle, or $CA = 4\,r$. Hence the circular arc AB and the cycloidal arc AD osculate at the point A. Consequently for small oscillations the bob of a simple pendulum moves approximately in the same path as that of a cycloidal pendulum, and therefore the motion through small arcs is approximately the same for both.

115. Brachistochrone. — One of the most famous problems in theoretical mechanics is to find the curve along which a particle will descend from one fixed point to another in the least possible time. This problem would arise, for example, if it was desired to find what form the return chutes in a bowling alley should have in order to return the balls to the players most quickly.

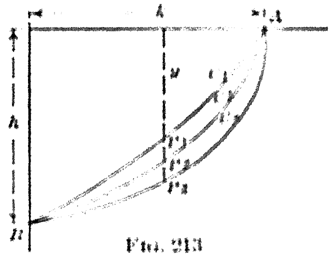


FIG. 213

Suppose, then, that A and B (Fig. 213) are the points in question, and let it be required to determine the curve of quickest descent between these points. Let C_1, C_2, C_3 be three curves of the same family through A and B, and assume that the time of descent along C_2 is less than it is along C_1 or C_3. In other words, assume that C_2 is the required curve, or brachistochrone, and let it be required to find its equation.

The velocity acquired by the particle at any point P on the curve at a depth y below the starting point is $v = \sqrt{2\,gy}$.

Since $v = \frac{ds}{dt}$, we have

$$dt = \frac{ds}{v} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2gy}}\, dx.$$

Therefore the total time of descent is

$$t = \frac{1}{\sqrt{2g}} \int_0^{x_2} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{y}}\, dx.$$

Now let this integral be denoted by I, and denote the derivative $\frac{dy}{dx}$ by y'. Then the integral I is a function of y and y'; namely,

$$I = \int \sqrt{\frac{1 + y'^2}{y}}\, dx,$$

which may be indicated by writing it in the form

$$I = I(y, y').$$

By assumption the curve C_2 is the curve of quickest descent; that is to say, the value of the integral I along C_2 is less than along either C_1 or C_3. To change the curve C_2 into C_1 or C_3 necessitates a change in the ordinates y and the slopes y', and hence in the integral I. Suppose, then, that y is changed by a finite amount δy, and y' by a finite amount $\delta y'$. Then I becomes a function of $y + \delta y$ and $y' + \delta y'$; namely, $I(y + \delta y, y' + \delta y')$. Expanding this by Taylor's theorem for two variables, we have

$$I(y + \delta y, y' + \delta y') = I(y, y') + \left[\delta y \frac{\partial}{\partial y} I(y, y') + \delta y' \frac{\partial}{\partial y'} I(y, y')\right]$$

$$+ \frac{1}{\underline{|2}}\left[\delta y^2 \frac{\partial^2}{\partial y^2} I(y, y') + 2\,\delta y \delta y' \frac{\partial^2}{\partial y \partial y'} I(y, y') + \delta y'^2 \frac{\partial^2}{\partial y'^2} I(y, y')\right] + \cdots,$$

or, if the first square bracket is denoted by δI, the second will be δI^2, etc., and hence the above expression may be written

$$I(y + \delta y, y' + \delta y') = I + \delta I + \frac{1}{\underline{|2}} \delta I^2 + \cdots.$$

Now since δI is a small quantity, each term of this series is smaller than the one preceding. Consequently the change in I will be positive or negative according as δI is positive or negative, unless $\delta I = 0$, in which case the change will be always positive, since a squared term, such as δI^2, is essentially positive. Since the curve sought is such that for this particular curve the integral I is less than for any neighboring curve of the same family, it is apparent that the condition which determines this curve is $\delta I = 0$, or from the definition of δI, this becomes

$$\delta I = \delta y \frac{\partial}{\partial y} I(y, y') + \delta y' \frac{\partial}{\partial y'} I(y, y') = 0.$$

Now the total change in the integral is equal to the sum of the changes in its several parts, or, expressed otherwise, the operations of variation and integration are commutative;* that is,

$$\delta I = \delta \int \sqrt{\frac{1 + y'^2}{y}}\, dx = \int \delta \sqrt{\frac{1 + y'^2}{y}}\, dx.$$

* In general, if any integral I is a function of an independent variable t and several dependent variables x, y, z, x', y', z', etc., say $I = \int \phi(t, x, y, z, x', y', z', \cdots)dt$, then by changing x to $x + \delta x$, y to $y + \delta y$, x' to $x' + \delta x'$, etc., and expanding both sides by Taylor's theorem, the result is

$$I + \delta I + \tfrac{1}{2}\delta^2 I + \cdots = \int_{t_0}^{t_1} (\phi + \delta\phi + \tfrac{1}{2}\delta^2\phi + \cdots)dt,$$

and consequently

$$\delta I = \int_{t_0}^{t_1} \delta\phi dt,$$

$$\delta^2 I = \int_{t_0}^{t_1} \delta^2\phi dt,$$

$$\cdots\cdots\cdots$$

$$\delta^n I = \int_{t_0}^{t_1} \delta^n\phi dt;$$

that is to say,

$$\delta^k \int_{t_0}^{t_1} \phi dt = \int_{t_0}^{t_1} \delta^k \phi dt,$$

or, expressed in words, the operations of variation and integration are commutative.

Furthermore, the symbol δ indicates the change in the function when the dependent variable is changed by a finite amount. Thus $\delta \frac{dq}{dt}$ indicates the change produced in $\frac{dq}{dt}$ when q is changed by the amount δq; that is to say,

$$\delta \frac{dq}{dt} = \frac{d(q + \delta q)}{dt} - \frac{dq}{dt} = \frac{d}{dt}(\delta q),$$

or, in words, the operations of variation and differentiation are also commutative.

Therefore, considering the function under the integral sign separately, we have

$$\delta\left[\sqrt{\frac{1+y'^2}{y}}\right] = \delta y\,\frac{\partial}{\partial y}\left(\sqrt{\frac{1+y'^2}{y}}\right) + \delta y'\,\frac{\partial}{\partial y'}\left(\sqrt{\frac{1+y'^2}{y}}\right)$$

$$= \delta y\left(-\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}}\right) + \delta y'\left(\frac{y'}{\sqrt{y(1+y'^2)}}\right).$$

Inserting this expression under the integral sign in the equation for t, then since $t = \frac{1}{\sqrt{2g}} I$ and consequently $\delta t = \frac{1}{\sqrt{2g}}\,\delta I$, we have

$$\delta t = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\left[\delta y\left(-\frac{\sqrt{1+y'^2}}{2\sqrt{y^3}}\right) + \delta y'\,\frac{y'}{\sqrt{y(1+y'^2)}}\right]dx.$$

Since the operations of variation and differentiation are also commutative we have $\delta y' = \delta\frac{dy}{dx} = \frac{d}{dx}\,\delta y$, and consequently the second term in the integral just obtained may be written

$$\frac{\frac{dy}{dx}}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\,\frac{d}{dx}\,\delta y.$$

To simplify this, apply to the function $\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\,\delta y$ the rule for the differentiation of a product. Then

$$\frac{d}{dx}\left\{\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\,\delta y\right\} =$$

$$\frac{d}{dx}\left\{\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\right\}\delta y + \frac{d}{dx}(\delta y)\left\{\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\right\}.$$

Transposing, and substituting the last term of this expression for the second term under the integral, the latter becomes

$$\delta t = \frac{1}{\sqrt{2g}}\int_0^k \left\{ \delta y\left[\frac{-\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{2y^{\frac{3}{2}}}\right] + \frac{d}{dx}\left[\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\,\delta y\right]\right.$$

$$\left. - \frac{d}{dx}\left[\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\right]\delta y \right\} dx.$$

The second term in this expression may be integrated at once since the integral and the derivative annul one another, leaving simply the quantity in brackets. Hence

$$\delta t = \frac{1}{\sqrt{2g}}\left\{\left[\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\,\delta y\right]_0^k\right. + \frac{1}{\sqrt{2g}}\int_0^k (\text{1st and 3d terms}).$$

Since the ends of the curves A and B are fixed, δy vanishes at both limits. Consequently the integrated term disappears at the upper limit. At the lower limit, however, the quantity y in the denominator is also zero, and hence the fraction takes the indeterminate form $\frac{0}{0}$. Evaluating this indetermination in the usual manner it is found to be zero,* and consequently the integrated

* By the ordinary rule for evaluating an indeterminate form, we have in this case to find the value of

$$\left[\frac{\frac{d}{dx}\left(\frac{dy}{dx}\,\delta y\right)}{\frac{d}{dx}\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}}\right]_{x=0}.$$

For the numerator we have

$$\left[\frac{d}{dx}\left(\frac{dy}{dx}\,\delta y\right)\right]_{x=0} = \left[\frac{dy}{dx}\,\delta\frac{dy}{dx}\right]_{x=0},$$

term completely disappears. Furthermore, in the two terms remaining under the integral sign, δy occurs as a factor. Since this is entirely arbitrary, if the integral δt is to be identically zero, the coefficient of δy must be zero. The condition $\delta t = 0$ therefore reduces to

$$-\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{2y^{\frac{3}{2}}} - \frac{d}{dx}\frac{\frac{dy}{dx}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}} = 0,$$

which is accordingly the differential equation of the required curve or brachistochrone.

Performing the differentiation indicated in the second term, this becomes

$$-\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{2y^{\frac{3}{2}}} - \frac{\frac{d^2y}{dx^2}}{\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}} + \frac{\left(\frac{dy}{dx}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}\right]}{2\left[y\left(1+\left(\frac{dy}{dx}\right)^2\right)\right]^{\frac{3}{2}}} = 0.$$

Reducing this expression to the common denominator

$$2\left[y\left(1+\left(\frac{dy}{dx}\right)^2\right)\right]^{\frac{3}{2}},$$

equating the resulting numerator to zero, and combining terms, the differential equation of the brachistochrone finally becomes

$$1+\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}=0,$$

and for the denominator

$$\left[\frac{d}{dx}\sqrt{y\left[1+\left(\frac{dy}{dx}\right)^2\right]}\right]_{y=0} = \left[\frac{1}{2}\frac{dy}{dx}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y}}\right]_{y=0}.$$

Hence the ratio of these two becomes

$$\left[2y\frac{dy}{dx}\sqrt{\frac{y}{1+\left(\frac{dy}{dx}\right)^2}}\right]_{y=0},$$

and this is zero since $y = 0$.

To integrate this equation, multiply through by $\frac{dy}{dx}$. Then it becomes

$$2\,y\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+\frac{dy}{dx}=0,$$

or, since
$$\frac{d}{dx}\left(y\left(\frac{dy}{dx}\right)^2\right)=\left(\frac{dy}{dx}\right)^3+2\,y\frac{dy}{dx}\frac{d^2y}{dx^2},$$

this reduces to
$$\frac{d}{dx}\left(y\left(\frac{dy}{dx}\right)^2\right)+\frac{dy}{dx}=0.$$

Integrating with respect to x, we have for the first integral

$$y\left(\frac{dy}{dx}\right)^2+y=c.$$

Writing this first integral in the form

$$\frac{dy}{dx}=\pm\sqrt{\frac{c-y}{y}},$$

and replacing $c-y$ by a new variable, say u, the equation takes the form

$$\frac{du}{dx}=\mp\sqrt{\frac{u}{c-u}}.$$

This, however, is the differential equation of a cycloid, as shown in the preceding article. Therefore in addition to the properties previously deduced, the cycloid is also the curve of quickest descent between any two given points.

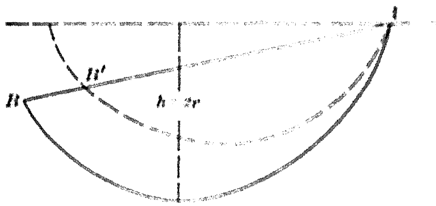


FIG. 214

It is of course possible to draw an infinite number of cycloids through any two given points. The particular one required by the solution, however, is that in which the constant of integration is $c=2\,r$, where r is the radius of the generating circle. To construct this particular cycloid, let A and B (Fig. 214) be the two given points. Then from the equation $\frac{dy}{dx}=\sqrt{\frac{c-y}{y}}$, it is evident that when $y=0$, $\frac{dy}{dx}=\infty$; that is to say, at the starting point A the tangent is vertical. Hence the

horizontal through A is the line on which the generating circle rolls. Now let the dotted curve in Fig. 214 represent any cycloid through A constructed on this horizontal line as base, and let B' denote its point of intersection with the line AB. Then since the required curve must be similar to this, it can be obtained by merely increasing the length of each chord AB', through the center of similarity A in the ratio $AB : AB'$. That is to say, all cycloids starting from the point A and whose generating circles roll on the horizontal through A constitute a family of similar and similarly situated curves, and the brachistochrone is that particular curve of the family which passes through the point B.

PROBLEMS

337. Two fixed points A and B have a difference in elevation of h feet. How long will it take a particle to move from A to B along the brachistochrone?

Solution. The time is given by the integral

$$t = \frac{1}{\sqrt{2g}}\int_0^h \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y}}\,dx = \frac{1}{\sqrt{2g}}\int_0^h \sqrt{\frac{1+\left(\frac{dx}{dy}\right)^2}{y}}\,dy.$$

From the equation of the brachistochrone,

$$\frac{dx}{dy} = \sqrt{\frac{y}{c-y}}.$$

Substituting this value in the above, we have for the time

$$t = \frac{1}{\sqrt{2g}}\int_0^h \sqrt{\frac{1+\frac{y}{c-y}}{y}}\,dy = \frac{1}{\sqrt{2g}}\int_0^h \sqrt{\frac{c}{cy-y^2}}\,dy,$$

whence by performing the integration,

$$t = \sqrt{\frac{c}{2g}}\,\text{vers}^{-1}\frac{2y}{c}\Big|_0^h = \sqrt{\frac{c}{2g}}\,\text{vers}^{-1}\frac{2h}{c}.$$

Since $c = 2r$, this becomes $t = \sqrt{\frac{r}{g}}\,\text{vers}^{-1}\frac{h}{r}$.

338. If the horizontal distance between the points A and B in the preceding problem is $k = 30$ ft. and their difference in elevation is $h = 3$ ft., find the shortest time of descent from A to B, and compare this with the time necessary to slide down the straight line joining the points.

Note. — Construct the brachistochrone through A and B, as explained in the preceding article. Then the depth of this curve below the horizontal through A, that is to say, its maximum ordinate, is $h = 2r$ (Fig. 214), from which r may be obtained. The other quantities necessary for finding t are given in the problem.

CHAPTER VI

KINETICS OF RIGID BODIES

116. Moment of Inertia. — In the preceding chapter, all the problems considered could be solved under the assumption that the mass of the body was concentrated at a single point. In most cases, however, especially those involving rotation, the actual shape of the body has an important effect upon its motion. Thus in the fundamental formula for rotation $T = I\alpha$ (see Chapter II, Art. 50), there appears the quantity I which depends upon the shape of the body, and was defined as

$$I = \sum mr^2.$$

This shape factor is called the **moment of inertia**, and as shown by its definition, is found by multiplying the mass of each particle by the square of its distance from the axis, or center of rotation, and taking their sum.

By considering a solid body as made up of an infinite number of material particles, the above finite summation becomes, in this case, an infinite summation or integration. The moment of inertia of a rigid body with respect to a point, line, or plane is therefore defined as $I = \lim\limits_{\Delta m \doteq 0} \sum r^2 \Delta m$, or simply

$$I = \int r^2 dm,$$

where r denotes the distance of any particle from the point, line, or plane in question, and the integration is to be extended throughout the body.

To illustrate the method for finding I, let it be required to find the moment of inertia of a slender uniform rod about an axis perpendicular to the rod and passing through its center (Fig. 215). For the element of mass in this case, take a section of the rod of

length dx. Then if A denotes the area of a cross section of the rod, the volume of this element is $A dx$. Consequently if δ denotes the density of the material, or weight per unit of volume, the total weight of the element considered is $A\delta dx$, and its mass is

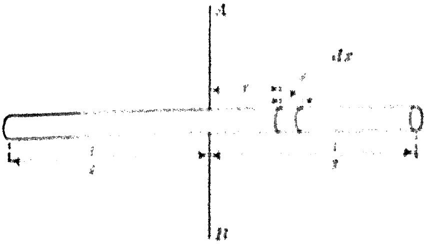


FIG. 215

$$dm = \frac{A\delta dx}{g}.$$

If, then, x denotes the distance of this element from the axis AB, we have $r = x$, and the value of I is

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{x^2\delta A dx}{g} = \frac{\delta A x^3}{3g}\Big|_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{\delta A l^3}{12 g}.$$

Since the mass M of the whole rod is $M = \frac{\delta A l}{g}$, this expression for the moment of inertia of the rod with respect to the axis AB may also be written

$$I = \frac{Ml^2}{12}.$$

As another illustration, let it be required to find the moment of inertia of a cylindrical disk of uniform thickness about an axis through its center and perpendicular to the plane of its face (Fig. 216). Since the figure is symmetrical about the axis, it will be simpler in this case to assume for the element of summation, or integration, a small ring of radius x and width dx. Let h denote the thickness of the disk. Then the element of volume is $2\pi x h dx$, and hence the element of mass is

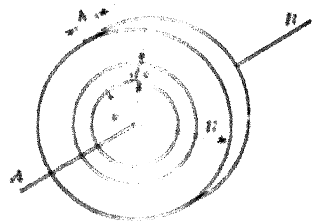


FIG. 216

$$dm = \frac{2\pi x h\delta dx}{g}.$$

Therefore the moment of inertia of the disk about the axis AB is

$$I = \int_0^R \frac{2\pi x h\delta dx}{g} x^2 = \frac{2\pi h\delta}{g} \frac{x^4}{4}\Big|_0^R = \frac{\pi h\delta R^4}{2g}.$$

Since the mass M of the entire disk is $M = \frac{\pi R^2 h\delta}{g}$, this expression for its moment of inertia about AB may be written

$$I = \frac{MR^2}{2}.$$

As a third illustration, let it be required to find the moment of inertia of a solid hemisphere about a diametral axis perpendicular to its plane face (Fig. 217). The simplest method in this case is to take for the element of integration a thin slice perpendicular to the axis of rotation, of radius r and thickness dx. From the results just obtained for a thin disk, the moment of inertia of such a slice about the axis AB through its center is $\frac{\delta\pi r^4 dx}{g}$. In this expression r is a variable, which varies as the ordinate to a semicircle of radius R. Thus from Fig. 217, $r^2 = R^2 - x^2$, and substituting this value of r in the expression for I, the moment of inertia of the entire hemisphere about AB becomes

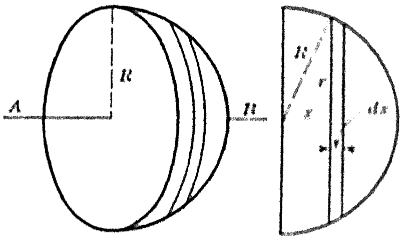


FIG. 217

$$I = \int_0^R \frac{\delta\pi}{g}(R^2 - x^2)^2 dx = \frac{\delta\pi}{g}\left(R^4 x - \frac{2}{3}R^2 x^3 + \frac{x^5}{5}\right)\Big|_0^R = \frac{8}{15}\frac{\pi\delta R^5}{g}.$$

Since the mass M of the entire hemisphere is $M = \frac{2}{3}\pi R^3 \frac{\delta}{g}$, its moment of inertia about AB may be written

$$I = \frac{4}{5} MR^2.$$

PROBLEMS

339. Find the moment of inertia of a triangular lamina of base b, altitude h, and thickness t about its base.

340. Find the moment of inertia of the same triangular lamina about a gravity axis parallel to its base.

341. Find the moment of inertia of a rectangular lamina with sides a and b and thickness t about the side b.

342. Find the moment of inertia of the rectangular lamina in the preceding problem about a gravity axis parallel to the side h.

343. Find the moment of inertia of a semicircular lamina of radius R and thickness t about its diametral chord.

344. Find the moment of inertia of the semicircular lamina in the preceding problem about the diameter perpendicular to its chord.

345. Find the moment of inertia of a sphere with respect to a diametral plane.

346. Find the moment of inertia of a sphere about a diameter.

347. Find the moment of inertia of a sphere with respect to its center.

348. Find the moment of inertia of a thin lamina in the form of a quadrant of a circle about one edge of the quadrant.

349. Find the moment of inertia of a thin lamina in the form of a segment of a circle about its chord.

117. Special Methods for Finding I. — In finding the moments of inertia of solids, it is often convenient to begin by finding I for a plane section of the solid. Strictly speaking, the I for a plane area is not a moment of *inertia*, since only solids have inertia. However, the definition of I for a plane figure is of precisely the same form as for a solid, and for this reason the same name is commonly applied to both.*

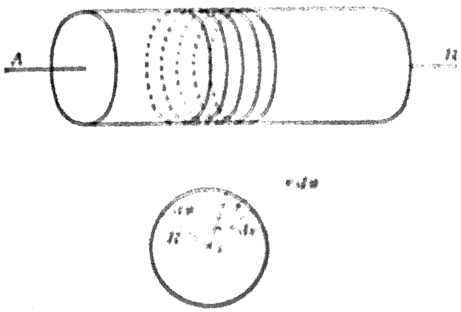


FIG. 218

To illustrate this method, let it be required to find the moment of inertia of a solid circular cylinder about its geometric axis (Fig. 218). Let R denote the radius of the cylinder and l its length. Begin by taking a plane section of the cylinder perpendicular to its axis, and finding I for the circle of radius R so obtained, with respect to its center. This may be solved like the problem of the circular disk in the preceding article, or as follows:

* The I for a plane area is sometimes called the "second moment of area," since it is defined as an area times the square of a distance.

Take polar coördinates, r and θ. Then since the element of mass dm is here replaced by an element of area dA, and since in polar coördinates $dA = rdrd\theta$, the expression for I becomes

$$I = \int_0^{2\pi}\int_0^{R} rdrd\theta \cdot r^2 = \frac{\pi R^4}{2}.$$

If A denotes the area of the circle, then $A = \pi R^2$, and hence

$$I = \frac{AR^2}{2}.$$

Now returning to the solid cylinder, the moment of inertia for a thin disk of thickness dx will be, from the result just obtained, $\frac{\pi R^4 dx}{2} \cdot \frac{\delta}{g}$, and hence for the entire cylinder

$$I = \int_0^{l} \frac{\pi R^4 dx}{2} \cdot \frac{\delta}{g} = \frac{\pi R^4 l \delta}{2g},$$

or, since the mass M of the cylinder is $M = \frac{\pi R^2 l \delta}{g}$, this becomes

$$I = \frac{MR^2}{2}.$$

As another illustration, let it be required to find the moment of inertia of a solid rectangular prism of dimensions a, b, c about an axis AB through its center and parallel to one edge (Fig. 219).

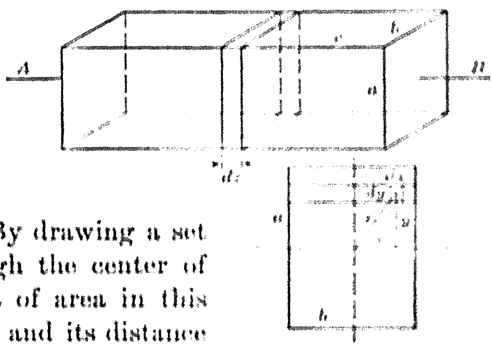


Fig. 219

Begin by taking a plane section perpendicular to the axis, and finding I for the rectangle so obtained about its center. By drawing a set of rectangular axes through the center of the rectangle, the element of area in this case may be taken as $dxdy$, and its distance from the center is $r = \sqrt{x^2 + y^2}$. Since there are two variables, x and y, involved, it is necessary to use double integration. Thus by one integration, say with respect to x, the I is obtained for an elementary strip of width dy, and by

the second integration with respect to y, this strip is extended to cover the entire figure. Hence

$$I = \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2)\,dx\,dy = \int_{-\frac{b}{2}}^{\frac{b}{2}}\left(\frac{a^3}{12} + ay^2\right)dy = \frac{a^3b}{12} + \frac{ab^3}{12}.$$

If A denotes the area of the rectangle, $A = ab$, and consequently

$$I = \frac{A}{12}(a^2 + b^2).$$

Returning to the solid body, the moment of inertia is the same for each plane section or thin lamina. Hence, taking a slice of thickness dz, we have

$$I = \int_0^c \frac{\delta}{g}\left(\frac{a^3b}{12} + \frac{ab^3}{12}\right)dz = \frac{\delta}{g}\left(\frac{a^3bc}{12} + \frac{ab^3c}{12}\right).$$

Since the mass M of the entire prism is $M = abc\frac{\delta}{g}$, we have finally for its moment of inertia with respect to the axis AB,

$$I = \frac{M}{12}(a^2 + b^2).$$

As a third example, find the moment of inertia of a circular cone about its geometric axis (Fig. 220). Begin, as before, by taking a plane section of the cone perpendicular to the axis AB. Since it has already been found that the I for a circle of radius x about its center is $I = \frac{\pi x^4}{2}$, we have by taking a thin lamina of thickness dy, and summing up for the entire cone,

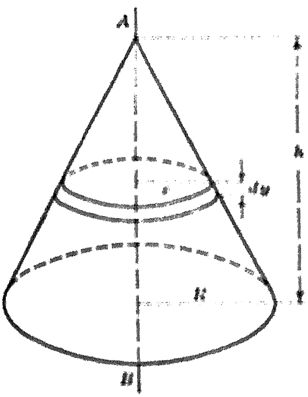


FIG. 220

$$I = \int \frac{\pi x^4}{2}\frac{\delta}{g}dy.$$

The radius x of the plane section is here directly proportional to the distance of this section from the vertex of the cone. Hence if the

vertex is taken as origin, $\frac{x}{R} = \frac{y}{h}$, and consequently

$$I = \int_0^h \frac{\pi}{2}\left(\frac{Ry}{h}\right)^4 \frac{\delta}{g}\,dy = \frac{\pi R^4 \delta y^5}{10\, h^4 g}\Big|_0^h = \frac{\pi R^4 \delta h}{10\, g}.$$

Since the mass M of the cone is $M = \frac{\pi R^2 h \delta}{3\, g}$, this becomes

$$I = \frac{3\, MR^2}{10}.$$

In general, for bodies having circular symmetry about a center or axis, the simplest method for finding I is to use polar coördinates, whereas if the body is symmetrical with respect to rectangular axes, rectangular coördinates should be used.

PROBLEMS

350. Find the moment of inertia of an ellipse of semiaxes a and b about each axis.

351. Find the moment of inertia of an ellipsoid of semiaxes a, b, c about the major axis a.

352. Find the moment of inertia of a rectangular pyramid about its geometric axis through the vertex and perpendicular to the base.

353. Find the moment of inertia of a hollow circular cylinder of external radius R and internal radius r about its geometric axis.

354. Find the moment of inertia of the hollow cylinder in the preceding problem about its equatorial axis, *i.e.* about a gravity axis perpendicular to its geometric axis.

355. Find the moment of inertia of a right triangular prism about a gravity axis perpendicular to its parallel bases.

118. Radius of Gyration. — As explained in Art. 116, the moment of inertia of a solid body with respect to a point, line, or plane is obtained by multiplying the mass of each infinitesimal element of the body by the square of its distance from the given point, line, or plane, as the case may be, and summing these products; that is, the moment of inertia I is defined as

$$I = \lim_{\Delta m \doteq 0} \sum r^2 \Delta m = \int r^2 dm.$$

Now suppose that the entire mass of the body is concentrated in a single material particle. Then a distance k may be found such that if this particle is placed at the distance k from the given axis (or point) it will have the same moment of inertia with respect to this axis as the body it represents. For this to be the case, it is only necessary that

$$Mk^2 = I,$$

or

$$k = \sqrt{\frac{I}{M}}.$$

The length k so defined is called the **radius of gyration**, and is of frequent application in practice.

The radius of gyration in any particular case may be found at once when the moment of inertia is known. Thus in Art. 116 the moment of inertia of a uniform rod of length l about an axis through its center was found to be $I = \frac{Ml^2}{12}$. Hence $\frac{Ml^2}{12} = Mk^2$, whence $k^2 = \frac{l^2}{12}$ and $k = \frac{l}{2\sqrt{3}}$.

Similarly for a circular disk about an axis through its center, $I = \frac{Mr^2}{2}$ and hence $k = \frac{r}{\sqrt{2}}$.

For a hemisphere about a diameter perpendicular to its base $I = \frac{2}{5} Mr^2$ and $k = r\sqrt{\frac{2}{5}}$.

For a cylinder about its geometric axis $I = \frac{Mr^2}{2}$ and $k = \frac{r}{\sqrt{2}}$, etc.

For a plane figure, the element of mass in the expression for I is replaced by an element of area, *i.e.*

$$I = \int r^2 dA.$$

Hence in finding the radius of gyration k of a plane figure, the moment of inertia must be equated to Ak^2, where A denotes the entire area. For example, the I for a circle was found in Art. 117 to be $I = \frac{Ar^2}{2}$, and equating this to Ak^2, we have $k = \frac{r}{\sqrt{2}}$.

Similarly for a rectangle ab about its center, it was found that $I = \frac{A}{12}(a^2 + b^2)$, and hence $k = \frac{\sqrt{a^2 + b^2}}{2\sqrt{3}}$.

PROBLEMS

356. Find the radius of gyration of a rectangle about a gravity axis parallel to one edge.

357. Find the radius of gyration of a sphere about a diametral plane.

358. Find the radius of gyration of an ellipsoid about its major axis.

359. Find the radius of gyration of a hollow circular cylinder about its equatorial axis, *i.e.* a gravity axis perpendicular to its geometric axis.

360. Find the radius of gyration of a rectangular rod of length l and sides a and b about its equatorial axis.

119. Theorems concerning Moments of Inertia. — I. *The sum of two rectangular moments of inertia of a plane area is equal to its corresponding polar moment of inertia.*

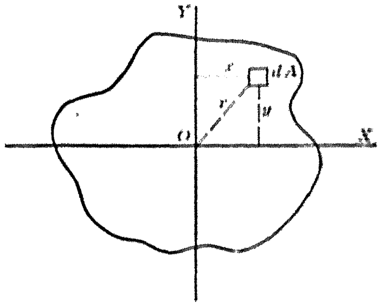


FIG. 221

PROOF: Consider the moments of inertia of a plane area about any two rectangular axes, such as OX, OY (Fig. 221). Then

$$I_x = \int y^2 dA, \quad I_y = \int x^2 dA.$$

The moment of inertia of the figure about any *point* (called a pole) such as O, Fig. 221, is known as the *polar moment of inertia*, and is defined as

$$I_p = \int r^2 dA.$$

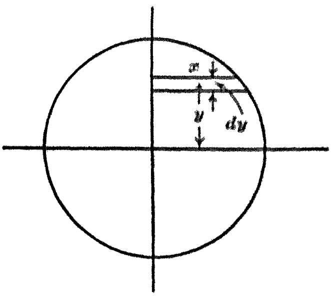


FIG. 222

Therefore since $x^2 + y^2 = r^2$, we have

$$I_p = I_x + I_y,$$

which proves the theorem.

To illustrate the theorem, let it be required to find the polar moment of inertia of a circle by calculating its moment of inertia with respect to a

diameter. In this case (Fig. 222)

$$I_x = 4\int_0^R x\,dy \cdot y^2 = 4\int_0^R \sqrt{R^2 - y^2}\, y^2 dy$$

$$= 4\left[\frac{y}{8}(2y^2 - R^2)\sqrt{R^2 - y^2} + \frac{R^4}{8}\sin^{-1}\frac{y}{R}\right]_0^R = \frac{\pi R^4}{4}.$$

Since a circle is symmetrical with respect to every diameter, $I_y = I_x$ and hence

$$I_p = I_x + I_y = \frac{\pi R^4}{2},$$

which agrees with the result obtained in Art. 117.

II. *The moment of inertia of a solid body with respect to a line is equal to the sum of its moments of inertia with respect to two planes which intersect at right angles in the line.*

The proof of this theorem is an obvious extension of the proof given for Theorem I, and is therefore not repeated.

III. *The moment of inertia of a solid body with respect to any axis is equal to its moment of inertia with respect to a parallel axis through its center of gravity, plus the mass of the body multiplied by the square of the distance between the two axes; or symbolically*

$$I = I_g + Mh^2,$$

where I = moment of inertia with respect to any axis, say AB, Fig. 223,

I_g = moment of inertia with respect to a gravity axis GG, parallel to AB,

h = distance between the axes AB and GG.

FIG. 223

PROOF: Let AB, Fig. 223, denote any given axis and GG a parallel axis through the center of gravity of the body. Then if r_1, r_2 denote the distances of any point P of the body from these two axes, respectively, its moments of inertia with respect to AB and GG are,

by definition, $$I = \int r_1^2 dm, \qquad I_g = \int r_2^2 dm.$$

Now by trigonometry, $r_1^2 = r_2^2 + h^2 - 2\, r_2 h \cos POX$, or, if x denotes the projection of r_2 on the perpendicular OX to the given axis, $r_2 \cos POX = x$, and hence

$$r_1^2 = r_2^2 + h^2 - 2\, hx.$$

Multiplying this equation through by the element of mass dm and integrating, we have

$$\int r_1^2 dm = \int r_2^2 dm + h^2 \int dm - 2\, h \int x dm.$$

In this expression $\int dm = M$, the total mass of the body, and $\int x dm = Mx_0$, where x_0 denotes the x coördinate of the center of gravity of the body referred to the axis GG. Therefore, since GG is a gravity axis, $x_0 = 0$, and the above relation becomes

$$I = I_g + Mh^2.$$

Since Mh^2 is a positive quantity, I is greater than I_g. Consequently of all moments of inertia about parallel axes, that about the axis through the center of gravity is the least.

To illustrate the theorem, let it be required to find the moment of inertia of a thin uniform rod or wire about an axis through one end and perpendicular to it. From Art. 116 the moment of inertia of the rod with respect to a parallel axis through its center is $I_g = \frac{Ml^2}{12}$; also h in the present case is $\frac{l}{2}$. Hence

$$I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}.$$

IV. ***The square of the radius of gyration with respect to any axis is equal to the square of the radius of gyration with respect to a parallel gravity axis plus the square of the distance between the axes.***

PROOF: Let k denote the radius of gyration of a solid body or plane area with respect to any given axis, and k_g the radius of gyration of the same body or area with respect to a gravity axis

parallel to the given axis, and let h denote the distance between these axes. From Theorem III we have

$$I = I_0 + Mh^2, \text{ or } Mk^2 = Mk_0^2 + Mh^2.$$

Hence
$$k^2 = k_0^2 + h^2,$$
which proves the theorem.

V. *If $k_1, k_2, \cdots k_n$ denote the radii of gyration of n bodies with respect to a given line or plane, the radius of gyration k of the entire system with respect to this line or plane is given by the relation*

$$k^2 = \frac{m_1k_1^2 + m_2k_2^2 + \cdots + m_nk_n^2}{m_1 + m_2 + \cdots + m_n}.$$

This theorem is useful in finding the moment of inertia of a composite body which can be conveniently separated into parts. The proof is obvious from the definitions of moment of inertia and radius of gyration.

VI. *Given the moments of inertia, I_x and I_y, of a plane area with respect to two rectangular axes OX and OY, one of which, say OX, is an axis of symmetry, the moment of inertia I, with respect to an axis in the same plane, passing through their intersection O, and inclined at an angle a to the axis of symmetry OX, is given by the relation*

$$I = I_x \cos^2 a + I_y \sin^2 a.$$

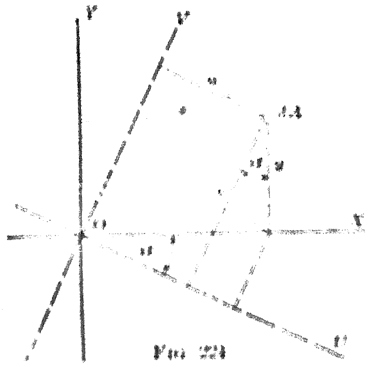


FIG. 224

PROOF: Let OU, Fig. 224, denote any line through the intersection O of the rectangular axes OX, OY, and inclined at an angle a to the axis of symmetry OX. Then the moment of inertia of the plane area with respect to OU is

$$I = \int v^2 dA = \int (y \cos a + x \sin a)^2 dA$$

$$= \int y^2 \cos^2 a\, dA + \int 2\, xy \sin a \cos a\, dA + \int x^2 \sin^2 a\, dA.$$

But since OX is assumed to be an axis of symmetry, each positive product term $(+y)\,x\,dA$ will have a like negative term $(-y)\,x\,dA$

to cancel it. Hence the second integral vanishes identically, and the relation becomes

$$I = \cos^2\alpha\int y^2dA + \sin^2\alpha\int x^2dA, \text{ or } I = I_x\cos^2\alpha + I_y\sin^2\alpha,$$

which proves the theorem.

PROBLEMS

361. Calculate the moment of inertia of a rectangular rod about an axis through its center and perpendicular to the rod.

362. Calculate the moment of inertia of a rod, elliptical in cross section, about an axis perpendicular to the rod and parallel to the short axis of the ellipse at a distance c from the center of the rod (Fig. 225).

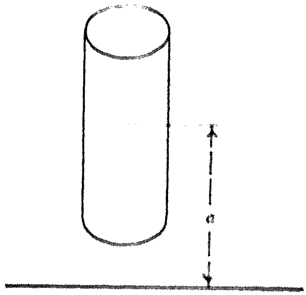


Fig. 225

363. A solid disk flywheel of cast iron is 12 in. in diameter and $2\frac{1}{2}$ in. thick. Find its moment of inertia about its axis.

364. The pendulum of a clock consists of a straight rod 3 ft. long and weighing $1\frac{1}{2}$ lb. and a lead disk 6 in. in diameter and weighing 5 lb., the center of the disk being at the end of the rod. Find the moment of inertia of the pendulum about an axis through the upper end of the rod, perpendicular to it and to the plane of the disk.

365. A cast-iron flywheel rim is 6 in. broad, 3 in. thick, and 5 ft. in external diameter. Find its moment of inertia and radius of gyration about its axis.

366. A cast-iron flywheel 15 ft. in external diameter has a rim 15 in. wide and 5 in. thick; hub 10 in. external diameter, 10 in. wide, and 4 in. bore; and six elliptical spokes 3 in. × 4 in. in section. Calculate the moment of inertia of the flywheel about its axis.

367. Find the moment of inertia of a truncated cone about its geometric axis.

368. Find the moment of inertia of a truncated pyramid about its geometric axis.

120. Inertia Ellipse. — Consider the moment of inertia of a plane area with respect to any line AB in its plane (Fig. 226). Let O be any point in this line, and OX, OY any pair of rectangular axes through O. Then if P denotes any point in the plane area under consideration, and α the angle between AB and OX, we have by geometry,

$$\overline{PC}^2 = \overline{PO}^2 - \overline{OC}^2,$$

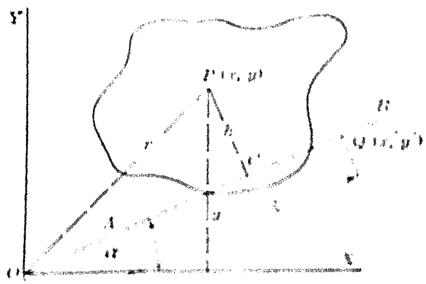


FIG. 226

or, if x, y denote the coördinates of P and h its perpendicular distance from AB, then

$$PC = h, \quad PO = \sqrt{x^2 + y^2},$$

$$OC = x \cos \alpha + y \sin \alpha,$$

and consequently, the above relation becomes

$$h^2 = x^2 + y^2 - (x \cos \alpha + y \sin \alpha)^2$$

$$= x^2(1 - \cos^2 \alpha) + y^2(1 - \sin^2 \alpha) - 2\,xy \sin \alpha \cos \alpha.$$

By means of the relation $\sin^2 \alpha + \cos^2 \alpha = 1$, this reduces to

$$h^2 = x^2 \sin^2 \alpha + y^2 \cos^2 \alpha - 2\,xy \sin \alpha \cos \alpha.$$

Multiplying this equation through by dA and integrating,

$$\int h^2 dA = \sin^2 \alpha \int x^2 dA + \cos^2 \alpha \int y^2 dA - 2 \sin \alpha \cos \alpha \int xy\,dA.$$

In this expression the first two integrals on the right are the moments of inertia of the plane area with respect to the Y and X axes, and the last integral is called a **product of inertia**, say P, where P is defined as

$$P = \int xy\,dA.$$

Inserting these values, the moment of inertia of the given plane area with respect to the axis AB becomes

$$I = I_x \cos^2 \alpha + I_y \sin^2 \alpha - 2\,P \sin \alpha \cos \alpha.$$

Now take a point Q on the line AB at an arbitrary distance k from O. Then the coördinates $x'y'$ of Q will be

$$x' = k \cos \alpha, \quad y' = k \sin \alpha,$$

Consequently if the above expression for I is multiplied throughout by k^2 and then expressed in terms of $x'y'$, it becomes

$$k^2 I = I_x k^2 \cos^2 \alpha + I_y k^2 \sin^2 \alpha - 2\,P k^2 \sin \alpha \cos \alpha$$

$$= I_x x'^2 + I_y y'^2 - 2\,P x' y'.$$

Since the distance k (or point Q) is arbitrary, let it be so chosen that $k^2I = 1$. Then this relation becomes

$$I_x x'^2 + I_y y'^2 - 2\,P x' y' = 1,$$

which represents an ellipse, since the coefficients I_x and I_y are both essentially positive. This is called the **inertia ellipse.**

Now it is shown in analytic geometry that for an ellipse there is always one pair of rectangular axes, namely, the axes of symmetry of the figure, such that the equation of the curve when referred to these axes contains no product terms. This pair of axes are called the **principal axes**, or major and minor axes. In the present case there is only one product term in the equation, and consequently only one axis need be an axis of symmetry so far as the elimination of this term is concerned.

Referring the inertia ellipse to its principal axes, therefore, the product term disappears, and the equation of the ellipse becomes

$$I_1 x'^2 + I_2 y'^2 = 1,$$

where I_1 and I_2 denote the moments of inertia of the given figure with respect to the new axes. The point Q, defined as above, lies on this ellipse. Moreover, since $k^2I = 1$, we have $I = \frac{1}{k^2}$, which means that if any radius vector OQ is drawn to the inertia ellipse, the moment of inertia of the figure with respect to this line as an axis is inversely proportional to the square of its length. Hence the moment of inertia of the figure is least with respect to the major axis OA of the inertia ellipse, and greatest with respect to its minor axis OB (Fig. 227). These results are expressed by the following theorem:

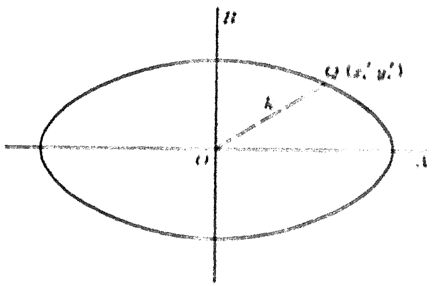


FIG. 227

For every plane area there can be found one, and only one, pair of rectangular axes, called principal axes, such that for these axes the product of inertia vanishes and the moment of inertia of the figure is a maximum with respect to one axis and a minimum with respect to the other.

121. Gyracion Ellipse. — In Theorem VI, Art. 119, it was shown that the moment of inertia of a plane area with respect to any axis of inclination α is expressed by the relation

$$I = I_x \cos^2 \alpha + I_y \sin^2 \alpha,$$

where I_x and I_y denote the moments of inertia with respect to a pair of rectangular axes of symmetry. It is evident from the preceding article, however, that an axis of symmetry must be a principal axis, since the product of inertia vanishes for such an axis.

Now divide through the above expression by the area A, and make use of the relation $k^2 = \frac{I}{A}$. Then it becomes

$$k^2 = k_x^2 \cos^2 \alpha + k_y^2 \sin^2 \alpha,$$

or, dividing through by k^2,

$$1 = \frac{k_x^2}{k^2} \cos^2 \alpha + \frac{k_y^2}{k^2} \sin^2 \alpha.$$

Now let each term be multiplied and divided by the same quantity, namely, $\frac{k_y^2}{k_y^2}$ for the first term and $\frac{k_x^2}{k_x^2}$ for the second term, which of course will not change its value since each fraction equals 1. Then

$$1 = \frac{k_x^2 k_y^2}{k^2} \cdot \frac{\cos^2 \alpha}{k_y^2} + \frac{k_x^2 k_y^2}{k^2} \cdot \frac{\sin^2 \alpha}{k_x^2},$$

or, if the common multiplier of each term is denoted by l^2, *i.e.* if $l^2 = \frac{k_x^2 k_y^2}{k^2}$, it simplifies into

$$1 = \frac{(l \cos \alpha)^2}{k_y^2} + \frac{(l \sin \alpha)^2}{k_x^2}.$$

If, then, a point (x, y) is chosen on the given axis at a distance l from the origin, we have

$$x = l \cos \alpha, \; y = l \sin \alpha,$$

and the above equation becomes

$$\frac{x^2}{k_y^2}+\frac{y^2}{k_x^2}=1,$$

which is the equation of an ellipse with semi-axes k_x and k_y. This ellipse, described on the principal radii of gyration as semi-axes, is therefore called the **gyration ellipse.**

The gyration ellipse has a certain geometrical property which makes it especially useful in practical calculations. Thus suppose that the gyration ellipse has actually been constructed to scale on the principal radii of gyration as semi-axes, and it is required to find the radius of gyration with respect to some other axis AB through the center O (Fig. 228). To find this new radius of gyration it is only necessary to draw a tangent to the ellipse parallel to AB, and the distance of this tangent from the center, measured to scale, will then be the required radius of gyration with respect to AB.

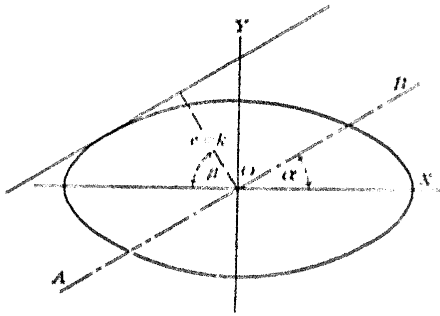


FIG. 228

To prove this theorem, let the equation of the gyration ellipse be $\frac{x^2}{k_y^2}+\frac{y^2}{k_x^2}=1$, as above, and let α denote the inclination of AB to OX (Fig. 228). The equation of a tangent to this ellipse at any point x', y' is

$$\frac{xx'}{k_y^2}+\frac{yy'}{k_x^2}=1, \text{ or } xx'k_x^2+yy'k_y^2=k_x^2k_y^2.$$

To express the equation of this straight line in the normal form, that is to say, in terms of the angle β which the normal to the tangent makes with OX, and the length of the perpendicular e, from the origin, it is only necessary to divide through by the sum of the squares of the coefficients of x and y. When so expressed, the equation becomes

$$\frac{xx'k_x^2}{\sqrt{x'^2k_x^4+y'^2k_y^4}}+\frac{yy'k_y^2}{\sqrt{x'^2k_x^4+y'^2k_y^4}}-\frac{k_x^2k_y^2}{\sqrt{x'^2k_x^4+y'^2k_y^4}}=0.$$

For simplicity let the radical in the denominators be denoted by R, *i.e.* let $R = \sqrt{x'^2k_x^4 + y'^2k_y^4}$. Then, as shown in analytic geometry,

$$\cos\beta = \frac{x'k_x^2}{R},\ \sin\beta = \frac{y'k_y^2}{R},\ e = \frac{k_x^2k_y^2}{R}.$$

Consequently, since $\beta = 90° - \alpha$, we have

$$k^2 = k_x^2\cos^2\alpha + k_y^2\sin^2\alpha = k_x^2\sin^2\beta + k_y^2\cos^2\beta$$

$$= k_x^2\frac{y'^2k_y^4}{R^2} + k_y^2\frac{x'^2k_x^4}{R^2} = \frac{k_x^2k_y^2(y'^2k_y^2 + x'^2k_x^2)}{R^2}.$$

Since $x'y'$ lies on the ellipse, its coördinates must satisfy the equation of the ellipse, *i.e.* $y'^2k_y^2 + x'^2k_x^2 = k_x^2k_y^2$. Consequently the expression for k^2 becomes

$$k^2 = \frac{k_x^4k_y^4}{R^2},$$

and by comparing this with the expression for e given above, it is evident that

$$k = e.$$

Hence the perpendicular distance from the origin of the tangent parallel to any line AB is equal to the radius of gyration corresponding to this line.

By plotting the gyration ellipse on each standard section in common use, it would thus be a simple matter to determine the radius of gyration with respect to any inclined axis.

122. Inertia Ellipsoid. — The results of Art. 120 may be extended so as to apply to solid bodies, as follows:

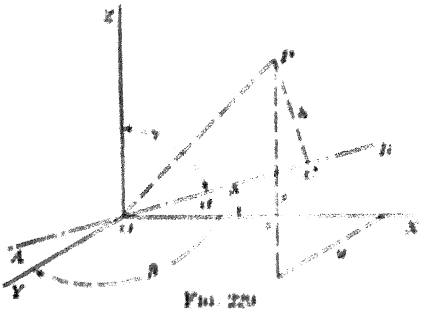


FIG. 229

Consider the moment of inertia of a solid body with respect to any given axis AB, let $O\ XYZ$ denote a set of rectangular axes through any point O of AB, and let P be any point of the body with coördinates x, y, z (Fig. 229). Drop a perpendicular PC from P on AB. Then if

l, m, n denote the direction cosines of AB, we have

$$OC = lx + my + nz.$$

Also if the length of the perpendicular PC is denoted by h, then

$$\begin{aligned} h^2 &= (x^2 + y^2 + z^2) - (lx + my + nz)^2 \\ &= x^2(1 - l^2) + y^2(1 - m^2) + z^2(1 - n^2) - 2(mnyz + lnzx + lmxy). \end{aligned}$$

Since the direction cosines of a line satisfy the relation $l^2 + m^2 + n^2 = 1$, we have $1 - l^2 = m^2 + n^2$, etc., and therefore the above expression may be written

$$\begin{aligned} h^2 &= x^2(m^2 + n^2) + y^2(l^2 + n^2) + z^2(l^2 + m^2) - 2(mnyz + lnzx + lmxy) \\ &= n^2(x^2 + y^2) + m^2(x^2 + z^2) + l^2(y^2 + z^2) - 2(mnyz + lnzx + lmxy). \end{aligned}$$

Multiplying this expression by the mass dm of the particle P and integrating, the result is

$$\begin{aligned} \int h^2 dm = n^2 \int (x^2 + y^2) dm + m^2 \int (x^2 + z^2) dm + l^2 \int (y^2 + z^2) dm \\ - 2\, mn \int yz\, dm - 2\, ln \int zx\, dm - 2\, lm \int xy\, dm. \end{aligned}$$

The first three terms on the right are the moments of inertia of the body with respect to the three coördinate axes, and will be denoted by A, B, C. The last three terms are called **products of inertia** and will be denoted by D, E, F. The definitions of moments and products of inertia are then as follows:

$$\text{Moments of inertia} \begin{cases} A = \int (y^2 + z^2)\, dm, \\ B = \int (x^2 + z^2)\, dm, \\ C = \int (x^2 + y^2)\, dm. \end{cases} \qquad \text{Products of inertia} \begin{cases} D = \int yz\, dm, \\ E = \int zx\, dm, \\ F = \int xy\, dm. \end{cases}$$

Using this notation, the expression for the moment of inertia I of the body with respect to the axis AB becomes

$$I = Al^2 + Bm^2 + Cn^2 - 2\, mnD - 2\, lnE - 2\, lmF.$$

Now take a point Q, with coördinates $x'y'z'$, on the line AB at an arbitrary distance k from the origin. Then

$$x' = kl, \quad y' = km, \quad z' = kn.$$

Therefore multiplying the expression for I by k^2 and expressing the result in terms of $x'y'z'$, it becomes

$$k^2 I = Ax'^2 + By'^2 + Cz'^2 - 2\,Dy'z' - 2\,Ex'z' - 2\,Fx'y'.$$

Since the distance k is arbitrary, it may be so chosen that $k^2 I = 1$, in which case this relation becomes

$$Ax'^2 + By'^2 + Cz'^2 - 2\,Dy'z' - 2\,Ex'z' - 2\,Fx'y' = 1.$$

Since this is an equation of the second degree, it represents some quadric surface, and since A, B, C are essentially positive, this surface must be an ellipsoid. This ellipsoid is known as **Poinsot's Central Ellipsoid.**

It is a property of an ellipsoid that there is always one, and only one, set of rectangular axes, called principal axes, for which the product terms vanish. When referred to its principal axes, therefore, the coefficients $D = E = F = 0$, and the equation of the ellipsoid reduces to $A_1x'^2 + B_1y'^2 + C_1z'^2 = 1.$

Moreover, since Q lies on the ellipsoid, and k is the length of the radius vector OQ (Fig. 230), it follows from the relation $I = \dfrac{1}{k^2}$ that the moment of inertia about any axis OQ is inversely proportional to the square of the radius vector to the ellipsoid, measured along this axis.

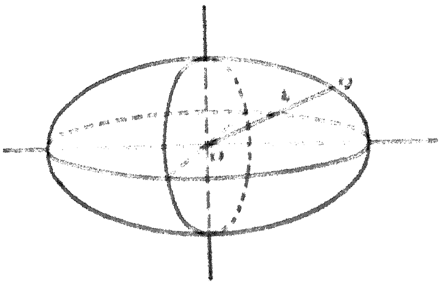


FIG. 230

As will be shown in Art. 174, the principal axes of a body have the property that if the center of gravity of the body lies on one of them, and the body is set in rotation about this axis, it will continue to rotate about it unless acted upon by an external moment. As will appear later, this property is of special importance in considering the kinetic reactions of rotating bodies.

123. Gyration Ellipsoid. — In Art. 121 the gyration ellipse was defined as an ellipse constructed on the two principal radii of gyration of any plane area as semi-axes. Extending this idea to

solid bodies, we obtain a **gyration ellipsoid**, defined as an ellipsoid constructed on the three principal radii of gyration of any solid body as semi-axes.

By the same process employed in Art. 121, the equation of the gyration ellipsoid is found to be

$$\frac{x^2}{k_{yz}^2}+\frac{y^2}{k_{xz}^2}+\frac{z^2}{k_{xy}^2}=1,$$

where k_{xy} denotes the radius of gyration of the body with respect to the line of intersection of the X and Y coördinate planes, etc.

By reasoning analogous to that given in Art. 121, it may be shown that the geometrical property of the gyration ellipse there deduced also holds for the gyration ellipsoid; namely, that

The radius of gyration, k, with respect to any line AB through the center O is equal to the perpendicular distance of the parallel tangent plane from O (Fig. 231).

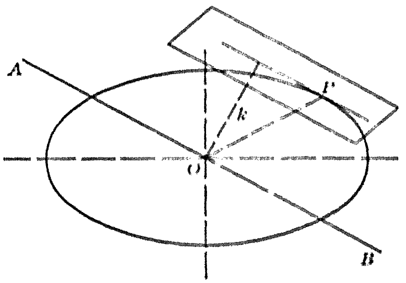


FIG. 231

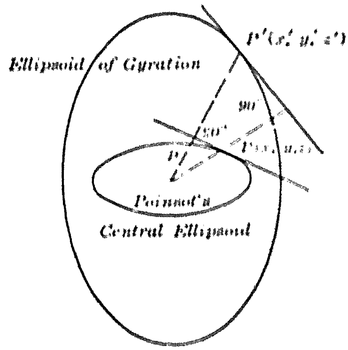


FIG. 232

124. Reciprocal Ellipsoids. — An important relation exists between Poinsot's central ellipsoid and the gyration ellipsoid, which will now be deduced.

Let the smaller ellipsoid in Fig. 232 represent Poinsot's central ellipsoid, the equation of which is

$$Ax^2+By^2+Cz^2=1,$$

and let P be any point on the surface with coördinates xyz. The equation of a tangent plane to this ellipsoid at the point P, in terms of the perpendicular p on this tangent plane and the direc-

tion cosines of the normal, is

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma,$$

where $R = \sqrt{A^2x^2 + B^2y^2 + C^2z^2}$, $\cos\alpha = \dfrac{Ax}{R}$, $\cos\beta = \dfrac{By}{R}$,

$$\cos\gamma = \frac{Cz}{R}, \quad p = \frac{1}{R},$$

and consequently $\cos\alpha = Axp$, $\cos\beta = Byp$, $\cos\gamma = Czp$.

Now on the perpendicular p to the tangent plane at P take a point P' such that the distance OP' is given by the relation

$$OP' = \frac{1}{p\sqrt{m}},$$

where m denotes the mass of the body. Then if $x'y'z'$ denote the coördinates of P', we have

$$x' = OP'\cos\alpha = \frac{1}{p\sqrt{m}}Axp = \frac{Ax}{\sqrt{m}},$$

$$y' = OP'\cos\beta = \frac{1}{p\sqrt{m}}Byp = \frac{By}{\sqrt{m}},$$

$$z' = OP'\cos\gamma = \frac{1}{p\sqrt{m}}Czp = \frac{Cz}{\sqrt{m}}.$$

From these relations between the coördinates of P and P', we have

$$x = \frac{x'\sqrt{m}}{A}, \quad y = \frac{y'\sqrt{m}}{B}, \quad z = \frac{z'\sqrt{m}}{C}.$$

If, then, these values of x, y, z are substituted in the equation of Poinsot's ellipsoid, namely,

$$Ax^2 + By^2 + Cz^2 = 1,$$

it becomes
$$\frac{x'^2m}{A} + \frac{y'^2m}{B} + \frac{z'^2m}{C} = 1.$$

This is the equation of another ellipsoid, having axes inversely proportional to the axes of Poinsot's ellipsoid. For this reason it is called the **Reciprocal Ellipsoid**.

Since $A = mk_{yz}^2$, $B = mk_{xz}^2$, $C = mk_{xy}^2$, where k_{yz} denotes the radius of gyration with respect to the line of intersection of the y and z coördinates planes, etc., we have

$$\frac{m}{A} = \frac{1}{k_{yz}^2}, \quad \frac{m}{B} = \frac{1}{k_{xz}^2}, \quad \frac{m}{C} = \frac{1}{k_{xy}^2},$$

and substituting these values in the equation of the reciprocal ellipsoid, it becomes

$$\frac{x'^2}{k_{yz}^2} + \frac{y'^2}{k_{xz}^2} + \frac{z'^2}{k_{xy}^2} = 1,$$

which represents the ellipsoid of gyration, as defined in Art. 123.

It is obvious that the converse of the relation here established is also true, namely, that Poinsot's central ellipsoid is reciprocal to the ellipsoid of gyration, and that to any point P' on the latter there corresponds a point P on Poinsot's ellipsoid such that OP is inversely proportional to OP'. The relation here established may be briefly expressed by saying that *Poinsot's ellipsoid and the gyration ellipsoid are mutually reciprocal.*

125. Physical Pendulum. — By a physical pendulum is meant a solid body of any shape or size which is suspended from a horizontal axis, not passing through its center of gravity, and allowed to swing under the action of gravity. The motion in this case is similar to that of the simple pendulum considered in Art. 112, the latter being in fact an ideal case of the motion under discussion.

Let G denote the center of gravity of the body, O a point on the axis of suspension, and ϕ the angular displacement of OG from the vertical (Fig. 233). The equation of motion is then simply

$$T = I\alpha,$$

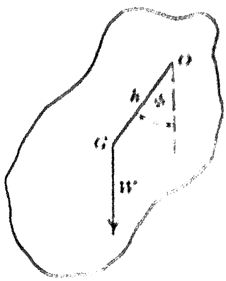


Fig. 233

where I denotes the moment of inertia of the body with respect to the given axis through O, α is its angular acceleration about this axis, and T is the external applied moment. Since this moment or torque is due in the present case simply to the weight of the body, it is equal to

$$T = Wh \sin \phi.$$

The equation of motion therefore becomes $Wh \sin\phi = I\alpha$, whence

$$\alpha = \frac{Wh \sin\phi}{I}.$$

For an ideal, or simple, pendulum, where the entire mass of the body is assumed to be concentrated at a single point at the distance l from the axis, the moment of inertia becomes $I = \frac{W}{g} l^2$, and the external moment is $T = Wl \sin\phi$. Consequently the equation of motion for the simple pendulum is $Wl \sin\phi = \frac{Wl^2\alpha}{g}$,

whence $$\alpha = \frac{g \sin\phi}{l}.$$

Equating these two expressions for α, we have

$$\frac{g \sin\phi}{l} = \frac{Wh \sin\phi}{I},$$

whence $$l = \frac{gI}{Wh}.$$

The value of l so obtained is called the *length of the equivalent simple pendulum*. That is to say, the period of oscillation of the actual physical pendulum is the same as for an ideal simple pendulum of length $l = \frac{gI}{Wh}$. Hence the expression for the period is

$$P = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{I}{Wh}}.$$

Now let k denote the radius of gyration of the body with respect to an axis through G and parallel to the axis of suspension. Then from theorem III, Art. 119, $I = \frac{W}{g}k^2 + \frac{W}{g}h^2$, and the equation of motion becomes

$$Wh \sin\phi = \frac{W}{g}(k^2 + h^2)\alpha.$$

The length of the equivalent simple pendulum may therefore be written as

$$l = \frac{gI}{Wh} = \frac{k^2 + h^2}{h} = h + \frac{k^2}{h}.$$

and the expression for the period becomes

$$P = 2\pi\sqrt{\frac{k^2 + h^2}{gh}}.$$

It is an important property of a physical pendulum that for every center of suspension O there is always a corresponding point O', called the **center of oscillation**, such that if the pendulum is suspended from this latter point it will oscillate with the same period as though suspended from O (Fig. 234). To prove this, consider a point O' on the line OG, and distant l from O, where l has the same value as above; that is,

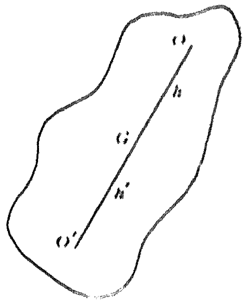


FIG. 234

$$OO' = l = h + \frac{k^2}{h}.$$

Then since $OG = h$, we have

$$O'G = h' = OO' - OG = \frac{k^2}{h},$$

and consequently $$OG \times O'G = k^2.$$

Hence if O' is made the center of suspension, O becomes the center of oscillation. That is to say, the centers of suspension and oscillation may be interchanged without affecting the period.

PROBLEMS

369. Show that for a physical pendulum the time of oscillation is a minimum when $h = k$.

370. A uniform rod 3 ft. long is suspended from a point 8 in. from one end. Find the time of a small oscillation.

371. A sphere 10 in. in diameter makes small oscillations about a horizontal tangent. Find the distance of the center of oscillation below the axis.

372. A circular disc 20 in. in diameter makes small oscillations about a horizontal tangent. Find the length of the equivalent simple pendulum.

126. Experimental Methods for Finding *I*. — The properties of the physical pendulum deduced in the last article offer a convenient means for finding I by experiment. Suppose, for example, that it is desired to find the moment of inertia of a wheel

which is of such a shape that I cannot be readily calculated. To find I in this case suspend the wheel from any convenient point, say a point under the rim, as shown in Fig. 235, and allow it to swing. The time of swing, or period, P, is then determined by taking the mean for any convenient number of complete oscillations. The weight of the wheel is also determined by weighing on an ordinary scales, and the distance h from the point of suspension to the center of gravity is measured. Then

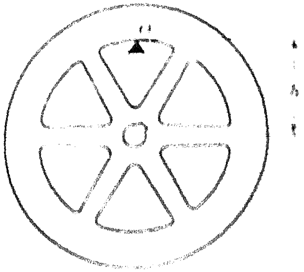


FIG. 235

by substituting these three measured quantities in the formula for the period, namely $P = 2\pi\sqrt{\frac{I}{Wh}}$, the moment of inertia I is found to be

$$I = \frac{WhP^2}{4\pi^2}.$$

The moment of inertia I_g of the wheel with respect to its gravity axis may then be found from the relation

$$I_g = I - \frac{Wh^2}{g}.$$

If a body is of such shape that it cannot be easily suspended, its moment of inertia may be found experimentally by means of a piece of apparatus called an oscillating table. This consists of a horizontal platform supported on a vertical axis to which is attached a spiral spring as shown in Fig. 236. When the table is rotated in either direction and then released, the spring will cause it to oscillate backward and forward about its vertical axis.

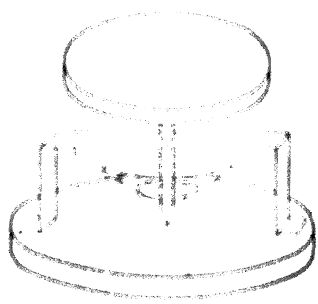


FIG. 236

Now let the period P of the apparatus when unloaded be determined by observation and then place a body of known weight W on the table at a certain distance d from the axis, and observe the period

P_1 of the whole apparatus when so loaded. Then since the period in each case is proportional to the square root of the moment of inertia, we have

$$\frac{P}{P_1} = \sqrt{\frac{I}{I + \frac{W}{g}d^2}},$$

whence
$$I = \frac{Wd^2}{g} \cdot \frac{P^2}{P_1^2 - P^2},$$

from which I may be calculated.

Having found I for the unloaded table in this way, the process may be reversed and the moment of inertia of a body of unknown weight or irregular shape found. Thus suppose I has been determined as above for the given apparatus and it is desired to find the moment of inertia I_0 of some irregular solid with respect to a given axis. Place this body on the table so that the given axis shall coincide with the vertical axis around which the table oscillates, and observe the period P_1 of the table when thus loaded. We have then the relation

$$\frac{P}{P_1} = \sqrt{\frac{I}{I + I_0}},$$

whence
$$I_0 = I\frac{P_1^2 - P^2}{P^2}.$$

Instead of supporting the table on a vertical axis, it may be suspended from its center of gravity by a wire. Such an arrangement is called a torsion pendulum. Evidently any lateral twist or angular displacement will produce a torsional strain in the wire and hence set up an oscillation as before. Let I denote the moment of inertia of the apparatus with respect to the axis of suspension, and suppose

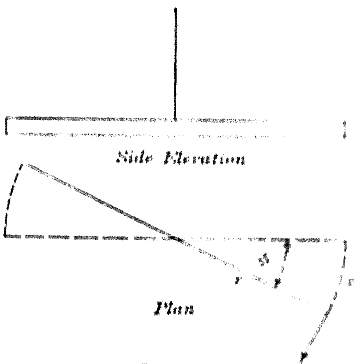


FIG. 237

that any given twisting moment T produces an angular displacement ϕ (Fig. 237). Then it is shown in the theory of torsion* that

$$\phi = \frac{Tl}{GI_0},$$

where l = length of suspending wire,

G = modulus of elasticity of shear,

I_0 = polar moment of inertia of a cross section of the wire.

Since the equation of motion is $T = I\alpha$, we have therefore

$$\alpha = \frac{T}{I} = \frac{\phi G I_0}{Il}.$$

Now denote the constant part of this expression by q; that is, let $q = \frac{GI_0}{Il}$. Then since $a = r\alpha$ and $x = r\phi$, we have

$$a = r\alpha = \frac{GI_0 r\phi}{Il} = qx.$$

In other words, the differential equation of motion is simply

$$\frac{d^2x}{dt^2} = qx,$$

and therefore, as in Art. 104, the expression for the period is

$$P = \frac{2\pi}{\sqrt{q}} = 2\pi\sqrt{\frac{Il}{GI_0}}.$$

If, then, it is required to find the moment of inertia of a torsion pendulum when its table is unloaded, place two equal and known weights W on the table at equal distances d from the axis so that they will be in static balance about the point of suspension. Then the sum of their moments of inertia with respect to the axis is $\frac{2\,Wd^2}{g}$, and the period of oscillation of the apparatus when so loaded will be

$$P_1 = 2\pi\sqrt{\frac{l\left(I + \frac{2\,Wd^2}{g}\right)}{GI_0}}.$$

* Slocum and Hancock, *Strength of Materials*, Revised Edition, p. 139.

Consequently, by division,

$$\frac{P}{P_1} = \sqrt{\frac{I}{I + \frac{2\,Wd^2}{g}}},$$

whence
$$I = \frac{2\,Wd^2}{g} \cdot \frac{P^2}{P_1^2 - P^2}.$$

Having once found I, the process may be reversed for finding the moment of inertia of any irregular solid, as previously explained for the oscillating table.

PROBLEMS

373. The period of vibration of an oscillating table when unloaded is 12 sec. By adding two weights of 1 lb. each at distances of 3 ft. from the axis, the period is increased to 13 sec. Find the moment of inertia of the unloaded table.

374. A flywheel is balanced upon a knife edge inside the rim and parallel to the axis of the wheel, at a distance of 3 ft. from the axis. If the period of vibration is observed to be 2.76 sec., find the radius of gyration of the wheel.

375. A metal disk is 12 in. in diameter and weighs 8 lb. It is suspended from its center by a vertical wire so that its plane is horizontal, and it is found that a twisting moment of 1 ft.-lb. will cause the wire to twist through 10°. When twisted and then released, how many oscillations will it make per minute?

376. A body is suspended by a vertical wire passing through its center of gravity, and it is found that 1 ft.-lb. of twisting moment produces a twist of 8°. When released it is observed to make 90 complete oscillations per second. Find its moment of inertia.

377. A flywheel weighing 4 T. is attached to a shaft, one end of which is fixed. The torsional rigidity of the shaft is such that it twists 0.5° for each foot-ton of torque applied to the flywheel. When twisted slightly and then released, the number of torsional vibrations is 125 per minute. Find the total moment of inertia of flywheel and shaft.

127. Impulsive Forces. — Suppose that a particle at rest is acted upon by an impulsive force; that is, a force suddenly applied, as in the case of a blow. Then the impulse given to the particle is measured by the momentum it acquires. For a system of particles constituting a solid body, the internal reactions equilibrate, and the external impulse communicated is therefore measured by the momentum acquired by the entire system or body.

The effect of an impulsive force acting on a body is, in general, to produce rotation about an instantaneous center or axis. Let O, Fig. 238, denote this center, G the center of gravity of the body, and $\bar{r}$ the distance between them. Then if F denotes the force acting on any particle of the body of mass m, v the velocity imparted to this particle, M the mass of the entire body, and ω its angular velocity, we have from the principle of impulse and momentum

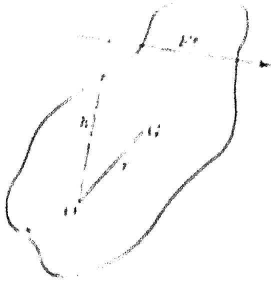


FIG. 238

$$\sum Ft = \sum mv = \sum mr\omega = \omega \sum mr = \omega M\bar{r}.$$

Moreover, by taking moments about O, we have as a second condition of equilibrium

$$\sum Fth = \sum mvr = \sum mr^2\omega = \omega \sum mr^2 = I\omega,$$

where I denotes the moment of inertia of the body about O. It is convenient to memorize this condition of equilibrium in the form

$$\text{angular velocity } \omega = \frac{\text{moment of impulse}}{\text{moment of inertia}}.$$

To illustrate the application of these formulas, suppose that a uniform rod receives an impulse perpendicular to it at some point C, and let O denote the instantaneous center about which it starts to revolve (Fig. 239). Then if W denotes the weight of the rod and v the velocity of its center of gravity G, we have

FIG. 239

$$Ft = \frac{Wv}{g},$$

and also
$$Fth = I\omega = \frac{Wk^2\omega}{g},$$

where k denotes the radius of gyration of the rod about O. Now the velocity of any point of the rod $= v \pm$ its velocity about G. Therefore since the velocity of the instantaneous center O is zero,

we have $$0 = v - \omega \cdot OG,$$

whence $$OG = \frac{v}{\omega}.$$

Since by division of the two conditions of equilibrium above we have $vh = k^2\omega$, this equation may be written

$$OG = \frac{k^2}{h} = \frac{k^2}{CG},$$

and consequently $$CG \times OG = k^2.$$

The point C is called the *center of percussion*, and the corresponding instantaneous axis through O the *axis of spontaneous rotation.*

If l denotes the length of the rod, then $k^2 = \frac{l^2}{12}$, and the distance between these two centers C and O is $\frac{2}{3}\,l$.

The center of percussion of a body is therefore that point at which it may be given an impulsive blow in a direction which is at right angles both to the axis of suspension and to the perpendicular let fall from the given point to the axis without exerting any impulsive action upon this axis.

PROBLEMS

378. A rod of length l and free to rotate about a fixed axis O receives an impulse perpendicular to the rod at a point C distant d from O (Fig. 239). Find the impulse on the axis and from this deduce the preceding results by assuming that O is the axis of spontaneous rotation, and therefore that the impulse on it is zero

379. At what point must a rod 3 ft. long be struck so that one end shall be initially at rest?

380. An armor plate suspended vertically is struck by a shot at a point vertically beneath its center of gravity and $\frac{1}{4}$ the width of the plate from the bottom. About what axis does the plate revolve?

381. Where should the stop be placed behind a door 7 ft. high and 2 ft. 8 in. wide so that there shall be no twist?

382. Why is the lighter end of a base ball bat held in the hand, and how would you determine at what point a ball should be struck with it so that the reaction on the hand shall be zero?

128. Lagrange's Equations. — There are certain general principles or theorems in mechanics, such as Lagrange's equations, Hamilton's principle, the principle of least work, and Gauss' principle of least constraint, which afford general solutions of certain types of

problems. Such general principles have therefore the advantage over ordinary methods in that once having found the general solution, any particular problem may be solved by merely routine processes. In this and the following articles, certain of these general principles are derived, and illustrated by practical applications.

Consider a system of rigid bodies so joined that only certain movements are possible. An example of such a system is a kinematic chain like the linkages shown in Figs. 162, 163, and 164. The position of all the members of such a system can be determined as soon as the position of one of them is known. Any quantity which determines the position of this one is therefore called a **generalized coordinate.**

In general, a system of rigid bodies, for example, a kinematic chain, has more than one degree of freedom, and hence requires more than one generalized coordinate to determine its position. Thus let n denote the number of degrees of freedom possessed by the system, and $q_1, q_2, \cdots q_n$, the corresponding set of generalized coordinates. The nature of these coordinates depends on the particular problem under consideration. For instance, if a body is constrained to move in a plane, its generalized coördinates may be taken as the radius vector and the azimuth; if constrained to lie on the surface of a sphere of given radius, the generalized coordinates may be taken as the longitude and colatitude, etc.

The rectangular coordinates x_e, y_e, z_e of any given point of the system are some definite functions of the n parameters or generalized coordinates $q_1, q_2, \cdots q_n$, say

$$x_e = x_e(q_1, q_2, \cdots q_n),$$
$$y_e = y_e(q_1, q_2, \cdots q_n),$$
$$z_e = z_e(q_1, q_2, \cdots q_n).$$

Differentiating each of these expressions with respect to the time t, we have

$$\frac{dx_e}{dt} = \frac{\partial x_e}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial x_e}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial x_e}{\partial q_n}\frac{dq_n}{dt},$$

or, if the generalized velocities $\frac{dq_1}{dt}$, etc., are denoted by q_1', etc.,

these first derivatives become

$$\frac{dx_r}{dt} = \frac{\partial x_r}{\partial q_1} q_1' + \frac{\partial x_r}{\partial q_2} q_2' + \cdots + \frac{\partial x_r}{\partial q_n} q_n',$$

$$\frac{dy_r}{dt} = \frac{\partial y_r}{\partial q_1} q_1' + \frac{\partial y_r}{\partial q_2} q_2' + \cdots + \frac{\partial y_r}{\partial q_n} q_n',$$

$$\frac{dz_r}{dt} = \frac{\partial z_r}{\partial q_1} q_1' + \frac{\partial z_r}{\partial q_2} q_2' + \cdots + \frac{\partial z_r}{\partial q_n} q_n'.$$

The kinetic energy E of the system, however, is given by the relation

$$E = \frac{1}{2} \sum {}_r m_r \left[\left(\frac{dx_r}{dt} \right)^2 + \left(\frac{dy_r}{dt} \right)^2 + \left(\frac{dz_r}{dt} \right)^2 \right],$$

where m_r denotes the mass of the rth particle and the summation is extended throughout the entire given system. To express E in terms of the generalized coördinates, substitute the expressions for $\frac{dx_r}{dt}$, $\frac{dy_r}{dt}$, $\frac{dz_r}{dt}$ just obtained. Then

$$E = \frac{1}{2} \sum {}_r m_r \left\{ \left[\frac{\partial x_r}{\partial q_1} q_1' + \frac{\partial x_r}{\partial q_2} q_2' + \cdots + \frac{\partial x_r}{\partial q_n} q_n' \right]^2 \right.$$
$$+ \left[\frac{\partial y_r}{\partial q_1} q_1' + \frac{\partial y_r}{\partial q_2} q_2' + \cdots + \frac{\partial y_r}{\partial q_n} q_n' \right]^2$$
$$\left. + \left[\frac{\partial z_r}{\partial q_1} q_1' + \frac{\partial z_r}{\partial q_2} q_2' + \cdots + \frac{\partial z_r}{\partial q_n} q_n' \right]^2 \right\}.$$

Now suppose that in any possible movement of the system, one of the generalized coördinates q_i is changed by an amount δq_i. Then the external forces acting on the system will, in general, do work. Let this element of work be denoted by

$$F_i \, \delta q_i,$$

where F_i may be called the generalized force corresponding to the coördinate q_i; that is to say, if q_i represents a distance, then F_i is an ordinary force, whereas if q_i is an angle, F_i represents the static moment of the external forces with respect to the axis of rotation.

A change δq_s in one of the parameters q_s, however, produces a corresponding change in any given rectangular coördinate x_r, y_r, z_r of amount δx_r, δy_r, δz_r, where

$$\delta x_r = \frac{\partial x_r}{\partial q_s}\delta q_s, \quad \delta y_r = \frac{\partial y_r}{\partial q_s}\delta q_s, \quad \delta z_r = \frac{\partial z_r}{\partial q_s}\delta q_s,$$

since $x_r = x_r(q_1, q_2 \cdots q_n)$ and only one of the parameters $q_1, q_2 \cdots q_n$ is assumed to undergo change. Hence equating the expression for the element of work in generalized coördinates to its value in rectangular coördinates, we have

$$F_s\delta q_s = \sum m_r\left[\frac{dx_r'}{dt}\delta x_r + \frac{dy_r'}{dt}\delta y_r + \frac{dz_r'}{dt}\delta z_r\right],$$

or, replacing δx_r, δy_r, δz_r by their values in terms of δq_s just obtained,

$$F_s\delta q_s = \sum m_r\left[\frac{dx_r'}{dt}\frac{\partial x_r}{\partial q_s} + \frac{dy_r'}{dt}\frac{\partial y_r}{\partial q_s} + \frac{dz_r'}{dt}\frac{\partial z_r}{\partial q_s}\right]\delta q_s,$$

and cancelling the common factor δq_s,

$$F_s = \sum m_r\left[\frac{dx_r'}{dt}\frac{\partial x_r}{\partial q_s} + \frac{dy_r'}{dt}\frac{\partial y_r}{\partial q_s} + \frac{dz_r'}{dt}\frac{\partial z_r}{\partial q_s}\right].$$

Now by the ordinary rule for the differentiation of a product, we have the relation

$$\frac{d}{dt}\left(x_r'\frac{\partial x_r}{\partial q_s}\right) = \frac{dx_r'}{dt}\frac{\partial x_r}{\partial q_s} + x_r'\frac{d}{dt}\left(\frac{\partial x_r}{\partial q_s}\right),$$

and using this identity in the expression for F_s just obtained, it becomes

$$F_s = \sum m_r\left\{\frac{d}{dt}\left(x_r'\frac{\partial x_r}{\partial q_s}\right) - x_r'\frac{d}{dt}\left(\frac{\partial x_r}{\partial q_s}\right) + \frac{d}{dt}\left(y_r'\frac{\partial y_r}{\partial q_s}\right)\right.$$

$$\left. - y_r'\frac{d}{dt}\left(\frac{\partial y_r}{\partial q_s}\right) + \frac{d}{dt}\left(z_r'\frac{\partial z_r}{\partial q_s}\right) - z_r'\frac{d}{dt}\left(\frac{\partial z_r}{\partial q_s}\right)\right\}.$$

From the expression for the energy E, obtained at the outset, we have by differentiating partially with respect to q_i,

$$\frac{\partial E}{\partial q_i}=\frac{1}{2}\sum_r m_r\frac{\partial}{\partial q_i}\left[\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right]$$

$$=\sum_r m_r\left[\frac{dx_r}{dt}\frac{\partial}{\partial q_i}\left(\frac{dx_r}{dt}\right)+\frac{dy_r}{dt}\frac{\partial}{\partial q_i}\left(\frac{dy_r}{dt}\right)+\frac{dz_r}{dt}\frac{\partial}{\partial q_i}\left(\frac{dz_r}{dt}\right)\right],$$

$$=\sum_r m_r\left[x_r'\frac{\partial}{\partial q_i}\left(\frac{dx_r}{dt}\right)+y_r'\frac{\partial}{\partial q_i}\left(\frac{dy_r}{dt}\right)+z_r'\frac{\partial}{\partial q_i}\left(\frac{dz_r}{dt}\right)\right],$$

or, since $$\frac{\partial}{\partial q_i}\left(\frac{dx_r}{dt}\right)=\frac{d}{dt}\left(\frac{\partial x_r}{\partial q_i}\right),\text{ etc.,}$$

this expression finally becomes

$$\frac{\partial E}{\partial q_i}=\sum_r m_r\left[x_r'\frac{d}{dt}\left(\frac{\partial x_r}{\partial q_i}\right)+y_r'\frac{d}{dt}\left(\frac{\partial y_r}{\partial q_i}\right)+z_r'\frac{d}{dt}\left(\frac{\partial z_r}{\partial q_i}\right)\right].$$

Furthermore, differentiating the second expression for E partially with respect to the generalized velocity q_i', we have

$$\frac{\partial E}{\partial q_i'}=\frac{1}{2}\sum_r m_r\frac{\partial}{\partial q_i'}\left\{\left[\frac{\partial x_r}{\partial q_1}q_1'+\frac{\partial x_r}{\partial q_2}q_2'+\cdots+\frac{\partial x_r}{\partial q_n}q_n'\right]^2+\cdots\right\}$$

$$=\sum_r m_r\left\{\left[\frac{dx_r}{dt}\right]\frac{\partial}{\partial q_i'}\left[\frac{\partial x_r}{\partial q_1}q_1'+\frac{\partial x_r}{\partial q_2}q_2'+\cdots+\frac{\partial x_r}{\partial q_n}q_n'\right]+\cdots\right\}$$

$$=\sum_r m_r\left[\frac{dx_r}{dt}\frac{\partial x_r}{\partial q_i}+\frac{dy_r}{dt}\frac{\partial y_r}{\partial q_i}+\frac{dz_r}{dt}\frac{\partial z_r}{\partial q_i}\right].$$

Consequently

$$\frac{d}{dt}\left(\frac{\partial E}{\partial q_i'}\right)=\sum_r m_r\frac{d}{dt}\left[\frac{dx_r}{dt}\frac{\partial x_r}{\partial q_i}+\frac{dy_r}{dt}\frac{\partial y_r}{\partial q_i}+\frac{dz_r}{dt}\frac{\partial z_r}{\partial q_i}\right]$$

$$=\sum_r m_r\frac{d}{dt}\left[x_r'\frac{\partial x_r}{\partial q_i}+y_r'\frac{\partial y_r}{\partial q_i}+z_r'\frac{\partial z_r}{\partial q_i}\right].$$

Subtracting from this relation the expression obtained above for $\frac{\partial E}{q_i}$, the result is the right member of the equation previously obtained for F_i. Therefore

$$F_i = \frac{d}{dt}\left(\frac{\partial E}{\partial q_i'}\right) - \frac{\partial E}{\partial q_i}.$$

This is Lagrange's equation for the coördinate q_i. Forming a similar equation for each coördinate, or degree of freedom, as many equations are obtained as unknowns, and their simultaneous solution therefore solves the problem.

It may be noted that Lagrange's method takes no account of internal work due to deformation, friction, etc.

129. Applications of Lagrange's Method. To illustrate the use of Lagrange's equations, they will first be applied to a simple piece of apparatus which is sometimes used to illustrate the conservation of angular momentum. This consists, as shown in Fig. 240, of a vertical spindle carrying a cross arm on which two equal weights slide in or out. When the arm revolves, the weights of course show a tendency to fly out, and if allowed to do so, the angular velocity of the spindle decreases, the angular retardation being of such amount that the angular momentum (or moment of momentum, as it is sometimes called) remains constant. On the other hand, if the weights are pulled in toward the center while the apparatus is in motion, its angular velocity is found to increase, the amount of the angular acceleration depending on how far and how rapidly the weights approach the center, the final velocity in any case being such as to give the same constant value to the angular momentum.

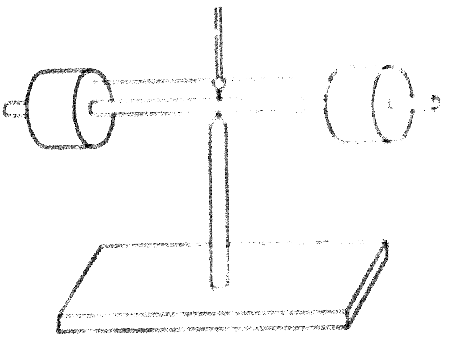


FIG. 240

In this piece of apparatus the generalized coördinates of the moving weights are their distance r from the axis of revolution

and their angular displacement ϕ about this axis, *i.e.*

$$q_1 = r, \qquad q_2 = \phi.$$

Moreover, the angular velocity is $\frac{d\phi}{dt} = \phi'$, and the velocity with which they approach, or recede from, the center is $\frac{dr}{dt} = r'$. Hence the total kinetic energy of either weight is

$$E = \tfrac{1}{2}\, m(r'^2 + r^2\phi'^2).$$

To apply Lagrange's method, first form the equation for the coördinate r. In the present case

$$\frac{\partial E}{\partial r} = mr\phi'^2,$$

$$\frac{\partial E}{\partial r'} = mr',$$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial r'}\right) = m\frac{dr'}{dt}.$$

If r' is constant, $\frac{dr'}{dt} = 0$ and Lagrange's equation for r is

$$F_r = \frac{d}{dt}\left(\frac{\partial E}{\partial r'}\right) - \frac{\partial E}{\partial r}$$

$$= -\, mr\phi'^2,$$

which is the ordinary expression for the centrifugal force, since ϕ' denotes the angular velocity ω.

To form the equation for the other coördinate ϕ, we have

$$\frac{\partial E}{\partial \phi} = 0,$$

$$\frac{\partial E}{\partial \phi'} = mr^2\phi',$$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \phi'}\right) = mr^2\frac{d\phi'}{dt} + 2\, mr\phi' r'$$

$$= I\alpha + 2\, mr\phi' r'.$$

Hence Lagrange's equation for ϕ is

$$F_\phi = \frac{d}{dt}\left(\frac{\partial E}{\partial \phi'}\right) - \frac{\partial E}{\partial \phi}$$

$$= I\alpha + 2\, mr\phi' r'.$$

If r is constant, the second term on the right disappears, and we have the ordinary equation for rotation; namely, $T = I\alpha$. Note that the generalized force F_ϕ for an angular coördinate is a torque. If r is not constant, then the term $2\, mr\phi' r'$ will not disappear, and hence in order to keep the angular velocity constant it will be necessary to apply a torque of this amount. If the weights slide out, r' is positive, and this torque must act in the same sense as $I\alpha$; if they slide in, r' is negative and the torque acts in the opposite sense to $I\alpha$.

A still simpler illustration of Lagrange's method is afforded by the motion of a physical pendulum (see Art. 125). In this case the body has only one degree of freedom, and hence there is but one generalized coördinate; namely, the angular displacement ϕ from the vertical. In this case $E = \frac{1}{2} I\phi'^2$, and consequently

$$\frac{\partial E}{\partial \phi} = 0, \quad \frac{\partial E}{\partial \phi'} = I\phi', \quad \frac{d}{dt}\left(\frac{\partial E}{\partial \phi'}\right) = I\frac{d\phi'}{dt}.$$

Hence Lagrange's equation for the only variable ϕ is

$$F_\phi = \frac{d}{dt}\left(\frac{\partial E}{\partial \phi'}\right) - \frac{\partial E}{\partial \phi}$$

$$= I\frac{d\phi'}{dt},$$

which is simply the fundamental equation $T = I\alpha$.

As a somewhat more complicated illustration, consider the motion of a bell and its clapper. This problem has a certain historical interest by reason of experiments performed with the Cologne "Kaiserglocke"* relative to the case in which a bell would not ring owing to the failure of the clapper to strike the side of the bell.

* Föppl, *Dynamik*, pp. 290 et seq.

Let O denote the point of suspension of the bell, O' the point of suspension of the clapper, W_1 the weight of the bell, W_2 the weight of the clapper, ϕ and ψ the angular displacements of bell and clapper from the vertical, and ϕ' and ψ' their corresponding angular velocities (Fig. 241). Since the bell has but one degree of freedom, its kinetic energy is simply

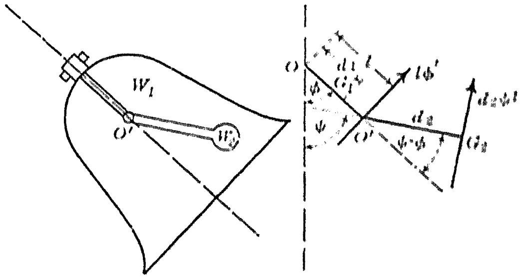


FIG. 241

$$E_1 = \tfrac{1}{2} I_1 \phi'^2.$$

The clapper, however, has two degrees of freedom since it swings about the point O' which is itself in motion. The kinetic energy of the clapper is therefore

$$E_2 = \tfrac{1}{2} I_2 \psi'^2 + \frac{W_2 v^2}{2g},$$

where v denotes the velocity of O'.

To find an expression for v, let l denote the distance OO', d_1 the distance from O to the center of gravity G_1 of the bell, and d_2 the distance from O' to the center of gravity G_2 of the clapper (Fig. 241). Then the linear velocity of O' relative to O is $l\phi'$, and the linear velocity of G_2 relative to O' is $d_2\psi'$. Hence the linear velocity v of G_2 relative to the fixed point O is given by the equation

$$v^2 = l^2\phi'^2 + d_2^2\psi'^2 + 2\, l\phi' d_2\psi' \cos(\psi - \phi).$$

Inserting this value of v^2 in the expression for E_2 and adding E_1, the total energy of bell and clapper in terms of the coördinates ϕ, ψ, ψ' becomes

$$E = \tfrac{1}{2} I_1\phi'^2 + \tfrac{1}{2} I_2\psi'^2 + \frac{W_2}{2g}(l^2\phi'^2 + d_2^2\psi'^2 + 2\, l d_2 \phi'\psi' \cos(\psi - \phi)).$$

Now obtain first the Lagrangian equation for ϕ. The derivatives in this case are

$$\frac{\partial E}{\partial \phi} = \frac{W_2}{g} l d_2 \phi' \psi' \sin(\psi - \phi),$$

$$\frac{\partial E}{\partial \phi'} = I_1 \phi' + \frac{W_2}{g} l^2 \phi' + \frac{W_2}{g} l d_2 \psi' \cos(\psi - \phi),$$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \phi'}\right) = I_1 \phi'' + \frac{W_2}{g} l^2 \phi'' + \frac{W_2}{g} l d_2 \psi'' \cos(\psi - \phi)$$

$$+ \frac{W_2}{g} l d_2 \psi' \sin(\psi - \phi)(\phi' - \psi'),$$

here ϕ'' denotes $\dfrac{d\phi'}{dt}$ and ψ'' denotes $\dfrac{d\psi'}{dt}$.

The equation for ϕ therefore becomes

$$F_\phi = \frac{d}{dt}\frac{\partial E}{\partial \phi'} - \frac{\partial E}{\partial \phi}$$

$$= \phi''\left(I_1 + \frac{W_2 l^2}{g}\right) + \psi'' \frac{W_2}{g} l d_2 \cos(\psi - \phi)$$

$$- \psi'^2 \frac{W_2}{g} l d_2 \sin(\psi - \phi).$$

Similarly, for the ψ equation we have

$$\frac{\partial E}{\partial \psi} = -\frac{W_2}{g} l d_2 \phi' \psi' \sin(\psi - \phi),$$

$$\frac{\partial E}{\partial \psi'} = I_2 \psi' + \frac{W_2}{g} d_2^2 \psi' + \frac{W_2}{g} l d_2 \phi' \cos(\psi - \phi),$$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \psi'}\right) = I_2 \psi'' + \frac{W_2}{g} d_2^2 \psi'' + \frac{W_2}{g} l d_2 \phi'' \cos(\psi - \phi)$$

$$+ \frac{W_2}{g} l d_2 \phi' \sin(\psi - \phi)(\phi' - \psi'),$$

and consequently

$$F_\psi = \frac{d}{dt}\left(\frac{\partial E}{\partial \psi'}\right) - \frac{\partial E}{\partial \psi}$$

$$= \psi''\left(I_2 + \frac{W_2}{g} d_2^2\right) + \phi'' \frac{W_2}{g} l d_2 \cos(\psi - \phi)$$

$$+ \phi'^2 \frac{W_2}{g} l d_2 \sin(\psi - \phi).$$

Since the generalized coördinates ϕ and ψ are angles, F_ϕ and F_ψ are torques.

Now these values of F_ϕ and F_ψ, obtained from the kinetic energy, must be equated to the corresponding moments of the external applied forces. The expressions for these external applied moments are found from the conditions of the problem. Thus if ϕ increases by a small amount while ψ remains constant, we have

$$F_\phi = W_1 d_1 \sin\phi + W_2 l \sin\phi,$$

whereas if ψ increases by a small amount while ϕ remains constant, we have

$$F_\psi = W_2 d_2 \sin\psi.$$

Equating these values of F_ϕ and F_ψ to those obtained from the kinetic energy, we have finally

$$\left.\begin{aligned}
\sin\phi(W_1 d_1 + W_2 l) &= \phi''\left(I_1 + \frac{W_2}{g} l^2\right) + \psi'' \frac{W_2}{g} l d_2 \cos(\psi - \phi) \\
&\qquad - \psi'^2 \frac{W_2}{g} l d_2 \sin(\psi - \phi), \\
\sin\psi \; W_2 d_2 &= \psi''\left(I_2 + \frac{W_2}{g} d_2^2\right) + \phi'' \frac{W_2}{g} l d_2 \cos(\psi - \phi) \\
&\qquad + \phi'^2 \frac{W_2}{g} l d_2 \sin(\psi - \phi).
\end{aligned}\right\}$$

Although these equations are so complicated as to be difficult to integrate, interesting conclusions may nevertheless be drawn from them without integration. Thus let it be required to determine the relation which must exist in order that the bell shall not ring.

In the first place, it is evident that the clapper will not strike the bell if their relative motion is not sufficiently great; that is to say, if $\psi - \phi$ does not exceed a certain limiting value, say γ. As a limiting case, suppose that this value is attained and

$$\psi - \phi = \gamma = \text{constant}.$$

Then $\frac{d\psi}{dt} = \frac{d\phi}{dt}$, *i.e.* $\psi' = \phi'$, and also $\frac{d\psi'}{dt} = \frac{d\phi'}{dt}$, or $\psi'' = \phi''$, and hence the equations of motion reduce to

$$\sin\phi\,(W_1d_1 + W_2l) = -\phi''\left(I_1 + \frac{W_2}{g}l^2 + \frac{W_2}{g}ld_2\cos\gamma\right) - \phi'^2\frac{W_2}{g}ld_2\sin\gamma.$$

$$\sin(\gamma + \phi)\,W_2d_2 = -\phi''\left(I_2 + \frac{W_2}{g}d_2^2 + \frac{W_2}{g}ld_2\cos\gamma\right) + \phi'^2\frac{W_2}{g}ld_2\sin\gamma.$$

If now the constants here involved can be so chosen that these equations are identical, it is evident that the system will move like a rigid body, and no relative motion take place between bell and clapper. The first condition for this is that the clapper shall lie in the axis of the bell, *i.e.* $\gamma = 0$, in which case the above equations further simplify into

$$\sin\phi\,(W_1d_1 + W_2l) = -\phi''\left(I_1 + \frac{W_2}{g}l^2 + \frac{W_2}{g}ld_2\right),$$

$$\sin\phi\,(W_2d_2) = -\phi''\left(I_2 + \frac{W_2}{g}d_2^2 + \frac{W_2}{g}ld_2\right).$$

Now these equations will be identical provided their coefficients are proportional, *i.e.* if the constants involved can be so chosen that the following relation is satisfied; namely,

$$\frac{I_1 + \frac{W_2}{g}l^2 + \frac{W_2}{g}ld_2}{W_1d_1 + W_2l} = \frac{I_2 + \frac{W_2}{g}d_2^2 + \frac{W_2}{g}ld_2}{W_2d_2}.$$

To obtain the physical meaning of this condition it may be observed that the weight of the clapper is small in comparison

with that of the bell. Under the assumption, then, that $I_2 = 0$ and W_2 is negligible in comparison with W_1, the condition becomes simply

$$\frac{I_1}{W_1 d_1} = \frac{d_2 + l}{g}.$$

Substituting for I_1 its value in terms of its radius of gyration k, with respect to an axis through the point of suspension O, namely $I_1 = \frac{W_1}{g} k^2$, this condition becomes

$$\frac{k^2}{d_1} = d_2 + l.$$

Or, if k_g denotes the radius of gyration of the bell with respect to an axis through its center of gravity G, we have $k^2 = k_g^2 + d_1^2$, and substituting this value, the above condition becomes

$$\frac{k_g^2}{d_1} + d_1 = d_2 + l.$$

The left member of this equality, however, is the distance from the point of suspension of the bell to its center of oscillation (see Art. 125), while the right member is the distance from the point of suspension of the bell to the center of gravity of the clapper. Therefore, if the clapper lies at the center of oscillation of the bell, the previous conditions will be fulfilled, and the bell will not ring.

130. Principle of the Conservation of Energy. — In Art. 101, Newton's laws of motion for a single particle of mass m were stated in the form of d'Alembert's principle, as expressed by the equations

$$m\frac{d^2x}{dt^2} = X,$$

$$m\frac{d^2y}{dt^2} = Y,$$

$$m\frac{d^2z}{dt^2} = Z.$$

From this simple statement of the laws of motion there may be derived the important result known as the **Principle of the Conserva-**

tion of Energy. To obtain this result, the first step is to multiply the first of the above equations by $\frac{dx}{dt}$, the second by $\frac{dy}{dt}$, and the third by $\frac{dz}{dt}$ and take their sum. Then

$$m\frac{d^2x}{dt^2}\frac{dx}{dt} + m\frac{d^2y}{dt^2}\frac{dy}{dt} + m\frac{d^2z}{dt^2}\frac{dz}{dt} = X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}.$$

This result applies to a single particle of mass m. To extend it to a rigid body, it is only necessary to consider the mass m infinitesimal, and sum throughout the body. Then integrating the expression with respect to t it becomes

$$\int_{t_1}^{t_2}\sum_e m_e\left[\frac{d^2x_e}{dt^2}\frac{dx_e}{dt} + \frac{d^2y_e}{dt^2}\frac{dy_e}{dt} + \frac{d^2z_e}{dt^2}\frac{dz_e}{dt}\right]dt$$

$$= \int_{t_1}^{t_2}\sum_e\left[X_e\frac{dx_e}{dt} + Y_e\frac{dy_e}{dt} + Z_e\frac{dz_e}{dt}\right]dt.$$

We have, however, the identity

$$\frac{d^2x}{dt^2}\frac{dx}{dt} = \frac{1}{2}\frac{d}{dt}\left(\frac{dx}{dt}\right)^2,$$

with similar relations for the y and z derivatives. Hence the integral on the left becomes

$$\frac{1}{2}\int_{t_1}^{t_2}\sum_e m_e\left[\frac{d}{dt}\left(\frac{dx_e}{dt}\right)^2 + \frac{d}{dt}\left(\frac{dy_e}{dt}\right)^2 + \frac{d}{dt}\left(\frac{dz_e}{dt}\right)^2\right]dt,$$

which may be written

$$\frac{1}{2}\int_{t_1}^{t_2}\sum_e m_e\frac{d}{dt}\left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2\right]dt,$$

and performing the indicated integration, this finally becomes

$$\sum_e \frac{m_e}{2}\left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2\right]_{t_1}^{t_2}.$$

Since the derivatives here involved represent the components of the velocity, i.e.

$$\frac{dx}{dt} = v_x,\ \frac{dy}{dt} = v_y,\ \frac{dz}{dt} = v_z \text{ where } v^2 = v_x^2 + v_y^2 + v_z^2,$$

the integrand becomes simply the kinetic energy, $\frac{1}{2}\sum_r m_r v_r^2$. Denoting the value of the kinetic energy at the limits t and t_0 by E and E_0 respectively, the above relation becomes

$$E - E_0 = \int_{t_0}^{t} \sum_r \left(X_r \frac{dx_r}{dt} + Y_r \frac{dy_r}{dt} + Z_r \frac{dz_r}{dt} \right) dt,$$

or, dividing out the differential dt,

$$E - E_0 = \int_{t_0}^{t} \sum_r \left(X_r dx_r + Y_r dy_r + Z_r dz_r \right).$$

Since the integral in the right member represents the work done in the given interval of time, this relation may be stated in words as follows:

The change in kinetic energy of a body in any given interval of time is equal to the work done on the body during this interval.

Any central field of force, such as that due to harmonic or Newtonian attraction, has the property that the rectangular components of the force of the field are the negative partial derivatives of a certain function of the coördinates called the **potential** of the field. That is, if X, Y, Z denote the rectangular components of the force at any point of the field and V denotes the potential of the field at this point, then

$$X_r = -\frac{\partial V}{\partial x_r}, \quad Y_r = -\frac{\partial V}{\partial y_r}, \quad Z_r = -\frac{\partial V}{\partial z_r}.$$

Since the expression for the total derivative of any function V in terms of its partial derivatives is

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz,$$

the potential is given by the integral

$$V = -\int X_r dx_r + Y_r dy_r + Z_r dz_r.$$

Any field of force possessing a potential function is called **conservative.** In a conservative field, therefore, the work done by the force of the field on a particle in moving from one point P_0

with coördinates x_0, y_0, z_0 to any other point P with coördinates x, y, z is

$$W = \int_{P_0}^{P} X_s dx_s + Y_s dy_s + Z_s dz_s$$

$$= -\int_{P_0}^{P} \frac{\partial V}{\partial x_s} dx_s + \frac{\partial V}{\partial y_s} dy_s + \frac{\partial V}{\partial z_s} dz_s = -(V - V_0),$$

where V and V_0 denote the values of the potential function at the points P and P_0 respectively. This relation, however, is independent of the path followed in passing from P to P_0. Consequently the work done in passing from one point of the field to another is the same for all paths between these points, and is equal to their negative difference in potential.

Equating this expression for the work to that previously obtained for the kinetic energy, the result is

$$E - E_0 = -(V - V_0).$$

Since this relation must be homogeneous in the units involved, the right member must have the dimensions of energy. The quantity $V - V_0$ or the change in the value of the potential of the field measured from an arbitrary point of reference P_0 is called the **potential energy**. Denoting the value of the potential energy by E_p, that is, putting $E_p = V - V_0$, and replacing the kinetic energy E by E_k, then since E_0 is a constant, the above relation becomes

$$E_k + E_p = \text{constant},$$

which is the energy equation for a conservative field, or the algebraic statement of the **principle of the conservation of energy**.

To illustrate the relation of a potential function to a conservative field, consider first the case of a constant field. Then

$$X = A, \quad Y = B, \quad Z = C,$$

where A, B, C denote constants, and

$$V = -\int A dx + B dy + C dz = -(Ax + By + Cz).$$

Hence, conversely,

$$X = -\frac{\partial V}{\partial x} = A, \quad Y = -\frac{\partial V}{\partial y} = B, \quad Z = -\frac{\partial V}{\partial z} = C.$$

For a harmonic field (see Art. 104)

$$X = -m\omega^2 x, \qquad Y = -m\omega^2 y, \qquad Z = -m\omega^2 z.$$

Consequently the potential of a harmonic field is

$$V = m\omega^2 \int (x\,dx + y\,dy + z\,dz) = \frac{m\omega^2}{2}(x^2 + y^2 + z^2),$$

and conversely $\qquad X = -\dfrac{\partial V}{\partial x} = -m\omega^2 x$, etc.

For a Newtonian field, or that due to the attraction of gravitation, the force of attraction or repulsion between any two particles is proportional to the product of the masses of the particles, and inversely proportional to the square of the distance between them; that is, the force F is given by

$$F = k\frac{mm'}{r^2},$$

where r denotes their distance apart and k is a factor of proportionality. The Newtonian potential is, therefore,

$$V = -\int k\frac{mm'}{r^2}\,dr = k\frac{mm'}{r}.$$

To obtain from the potential by differentiation the components of the force, the potential must first be expressed in terms of the rectangular coördinates of the particles considered. Thus if one particle is of unit mass, and the other of mass m and density γ, we have

$$V = \int^z\!\!\int^y\!\!\int^x \frac{k\gamma\,dx\,dy\,dz}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}},$$

where x, y, z denote the coördinates of one particle, x', y', z' those of the other, and $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ is their distance apart. From this expression

$$X = -\frac{\partial V}{\partial x} = \int^z\!\!\int^y\!\!\int^x \frac{k\gamma(x-x')\,dx\,dy\,dz}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{3}{2}}}$$

$$= \int^z\!\!\int^y\!\!\int^x \frac{k\gamma(x-x')\,dx\,dy\,dz}{r^3},$$

with similar expressions for Y and Z. To verify this expression for X, note that

$$\int\int\int \frac{k\rho\,dx\,dy\,dz}{r^2} = F \text{ and } \frac{x - x'}{r} = \cos\alpha,$$

and consequently the integral represents simply

$$X = F\cos\alpha.$$

The work done in carrying a particle of mass m along any path against the force of mutual attraction $F = k\frac{mm'}{r^2}$ exerted between it and another particle of mass m' from a distance apart r_1 to the distance r_2 is

$$W = \int_{r_1}^{r_2} k\frac{mm'}{r^2}\,dr = -\left(k\frac{mm'}{r_2} - k\frac{mm'}{r_1}\right) = V_1 - V_2.$$

By adding and subtracting the potential V_0 at some fixed point of the field, this may be written

$$W = (V_1 - V_0) - (V_2 - V_0).$$

Consequently the work done is equal to the change in potential energy of the system, and by the principle of the conservation of energy, this is also equal to the change in kinetic energy. Work therefore represents energy in transit from one form to another, as explained in Art. 43.

In the case of a body near the earth and acted on by gravity only, the force of attraction depends on its distance from the center of the earth. Hence the difference in potential energy of the body in any two positions depends only on their vertical distance apart. Therefore the difference in kinetic energy in these two positions also depends only on their vertical distance apart, from which it follows that the velocity of a falling body at any given instant depends only on the vertical height fallen.

PROBLEMS

383. Determine the gravitation potential at the surface of the earth and the attraction of gravitation at this point.

SOLUTION. It has been found by experiment that the acceleration of a body falling freely at the surface of the earth is 981 cm/sec². Also from a

study of the figure of the earth and determinations of its density, its mass has been found to be $m = 6.14 \times 10^{27}$ grams, and its mean radius $r = 6.37 \times 10^8$ cm.

Hence if a mass of m' grams is acted on by the earth, we have by comparing the expressions $F = m'a$ with $F = k\frac{mm'}{r^2}$,

$$981\, m' = k\frac{6.14 \times 10^{27}\, m'}{(6.37 \times 10^8)^2},$$

whence
$$k = \frac{1}{1.543 \times 10^7}.$$

Suppose, however, that it is required to determine what mass must be condensed into a particle in order that its attraction on an equal mass distant 1 cm. may be one dyne. In this case if m denotes the required mass in grams, we have

$$1 = \frac{m \times m}{1.543 \times 10^7},$$

whence $m = 3928$. This is called the *astronomical unit of mass*, because if masses are expressed in this unit, the equation of attraction becomes

$$F = \frac{mm'}{r^2}.$$

In the present case the mass of the earth expressed in astronomical units is

$$m = \frac{6.14 \times 10^{27}}{3928}.$$

Hence the potential at the earth's surface is

$$V = \frac{m}{r} = \frac{6.14 \times 10^{27}}{3928 \times 6.37 \times 10^8} = 2.454 \times 10^{15} \text{ ergs}.$$

This represents the work that must be applied to each unit of mass to carry it from the earth's surface to an infinite distance against the attraction of gravitation.

384. A bullet weighing 1 oz. and moving at 1600 ft./sec. is fired into a block of wood weighing 5 lb., which is free to move. How much energy is lost in the impact?

(Note that momentum is conservative, but mechanical energy is not.)

385. A shell weighing 10 lb. bursts into two pieces weighing 4 and 6 lb., which move on with velocities of 1200 ft./sec. and 800 ft./sec. respectively. Find the velocity of the shell before bursting and the energy gained from the explosion.

131. Principle of Least Work. — In Chapter III it was shown that most problems involving the composition and resolution of forces could be solved by applying the ordinary conditions of equilibrium, namely Σ forces $= 0$ and Σ moments $= 0$. There are certain prob-

lems in statics, however, which these conditions alone are insufficient to solve. That is to say, the number of unknown quantities involved is greater than the number of relations furnished by the ordinary conditions of equilibrium. Such problems are called *statically indeterminate*. To solve problems of this nature it is necessary to introduce some general principle which will in all cases furnish as many equations of condition as there are unknowns. Such a general method is afforded by the **Principle of Least Work**, which in this respect ranks with Hamilton's principle and Lagrange's method as one of the most powerful methods of mechanics.

In Article 77, Chapter III, it was shown that the condition for stable equilibrium of a system having one degree of freedom is that its potential energy shall be a minimum. Since in all statical problems we are concerned solely with stable equilibrium, the condition that the potential energy must be a minimum evidently applies. Now suppose that the potential energy is expressed in terms of the unknown quantities which it is desired to determine. Then the condition that this expression shall be a minimum resolves itself into the condition that the partial derivatives of the potential energy with respect to each of the unknowns involved shall be zero. In this way we obtain exactly as many equations as unknowns, from which these unknown quantities may be found. Thus if V denotes the potential energy and $F_1, F_2, \cdots F_n$ the unknown quantities to be found, first express V as a function of these unknowns, say $V(F_1, F_2, \cdots F_n)$. Then the condition for a minimum is $dV = 0$, or expressing dV as a total differential,

$$dV = \frac{\partial V}{\partial F_1} dF_1 + \frac{\partial V}{\partial F_2} dF_2 + \cdots + \frac{\partial V}{\partial F_n} dF_n = 0.$$

Since $F_1, F_2, \cdots F_n$ are assumed to be independent, in order for this relation to be satisfied identically, the coefficients of $dF_1, dF_2 \cdots dF_n$ must all be zero; that is,

$$\frac{\partial V}{\partial F_1} = 0, \quad \frac{\partial V}{\partial F_2} = 0, \cdots \frac{\partial V}{\partial F_n} = 0.$$

We have thus n equations from which to determine the n unknowns $F_1, F_2, \cdots F_n$.

In the case of elastic solids, the effect of the external forces is to produce deformation. In producing such deformation, the external forces do work upon the body, called work of deformation. If the elastic limit of the material is not passed, this work of deformation is stored up as potential energy, which is given out again when the external forces are removed. In this case the above condition takes the form that *for stable equilibrium the work of deformation is a minimum.* This condition is known as the **Principle of Least Work.**

Before applying the principle it will be necessary to show how to find the work of deformation, or potential energy, for elastic solids subjected to direct stress and to bending stress.

I. Direct Stress. Consider a straight bar of length l and cross section A. It is found by experiment that in elastic solids we have the relation

$$\frac{\text{unit stress}}{\text{unit deformation}} = \text{constant},$$

provided the stress does not pass the elastic limit of the material. This is called Hooke's law. If p denotes the unit stress and P the total load on the bar, then $p = \frac{P}{A}$. Also, if s denotes the unit deformation and Δl the total deformation, or change in length of the bar, then $s = \frac{\Delta l}{l}$. Denoting the constant ratio above by E, called Young's modulus, Hooke's law becomes

$$\frac{p}{s} = E,$$

or, replacing p and s by their values as just given,

$$\frac{Pl}{\Delta l A} = E, \text{ whence } \Delta l = \frac{Pl}{AE}.$$

Therefore, since when the load is applied it gradually increases from zero up to its full amount P, the average force acting on the bar during the deformation is $\frac{1}{2} P$, and consequently the work of deformation is

$$V = \frac{P^2 l}{2\,AE}.$$

II. Bending Stress. To find the work of deformation for a prismatic beam subjected to a bending moment T, consider two adjacent cross sections of the beam at a distance dx apart (Fig. 243). Let $d\beta$ denote the angular change between the planes of the cross sections, produced by the bending. Then neglecting infinitesimals which for ordinary curvature are of order higher than the first, the change in length of a fiber at a distance y from the neutral axis is $y d\beta$, and hence by Hooke's law

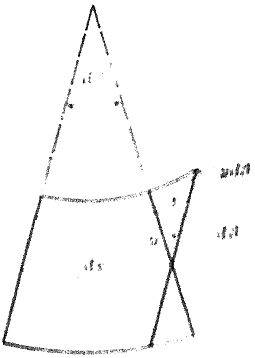


FIG. 243

$$\frac{y d\beta}{dx} = \frac{p}{E}.$$

It is a fundamental principle in the theory of beams, however, that the stress varies as the distance from the neutral axis, *i.e.* $p = \frac{Ty}{I}$, where I denotes the static moment of inertia of the cross section with respect to the neutral axis. Hence substituting this value of p in the above, the angular deformation $d\beta$ becomes

$$d\beta = \frac{T dx}{EI}.$$

Since the external moment T is zero when first applied and gradually increases up to its full value, its average value is $\frac{1}{2}T$. Therefore the work of deformation for the given element of length dx is

$$dW = \tfrac{1}{2} T d\beta = \frac{T^2 dx}{2EI},$$

and, consequently, for the entire beam

$$V = \int_0^l \frac{T^2 dx}{2EI}.$$

The principle of least work will now be applied to a variety of simple problems, which are otherwise statically indeterminate.

PROBLEMS

386. Two short posts of the same length l, but of cross-section areas A_1 and A_2, and of material having moduli E_1 and E_2 respectively, carry a load P jointly (Fig. 243). How much of the load is carried by each?

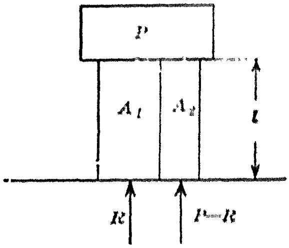


FIG. 243

SOLUTION. Let R denote the load carried by No. 1. Then the load carried by No. 2 is $P - R$. Hence, applying the expression for the work due to a direct stress, the total work of deformation for both posts is

$$V = \frac{R^2 l}{2\,A_1E_1} + \frac{(P - R)^2 l}{2\,A_2E_2}.$$

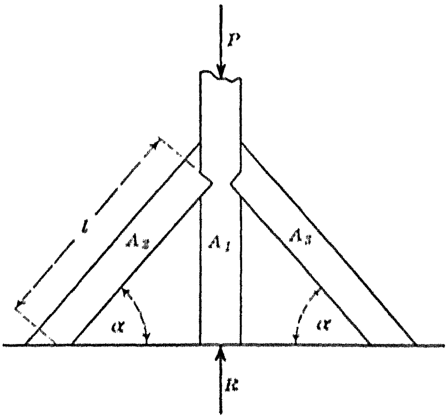


FIG. 244

The condition for a minimum gives

$$\frac{dV}{dR} = 0 = \frac{Rl}{A_1E_1} - \frac{(P - R)\,l}{2\,A_2E_2},$$

whence,

$$R = \frac{A_1E_1}{A_1E_1 + A_2E_2}P.$$

387. A post supporting a load P is braced at the bottom by two braces each of length l and inclined at the same angle α to the horizontal (Fig. 244). If the upright is of cross section A_1 and has a modulus E_1, and the braces are each of cross section A_2 and modulus E_2, show that the load R carried by the upright BD is given by

$$R = \frac{A_1E_1}{2\sin^3\alpha A_2E_2 + A_1E_1}P.$$

388. A platform 12 ft. × 18 ft. in size and weighing 1 ton is supported at the corners by four wooden legs, each 8 in. square. A load of 5 tons is placed on this platform 4 ft. from each of two adjoining edges (Fig. 245). How much of the load is carried by each leg?

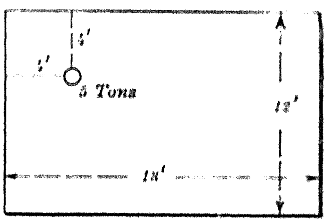


FIG. 245

389. A uniformly loaded beam of length $2\,l$ is supported at its center and ends. Find the reactions of the supports.

SOLUTION. Let R denote the middle reaction and P_1, P_2 the two end reactions (Fig. 216). Then from symmetry $P_1 = P_2$, and also $P_1 = wl - \frac{R}{2}$. Hence there is but one unknown, say R, to be determined. The moment at any point distant x from the left support is

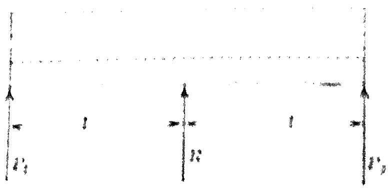


FIG. 216

$$T = P_1 x - \frac{wx^2}{2},$$

and consequently the work of deformation for the left half of the beam is

$$\int_0^l \frac{T^2 dx}{2EI} = \frac{1}{2EI}\left[\frac{P_1^2 l^3}{3} - \frac{P_1 w l^4}{4} + \frac{w^2 l^5}{20}\right].$$

Since the work of deformation for the other half of the beam is of the same amount, the total work of deformation is

$$V = \frac{1}{EI}\left[\frac{P_1^2 l^3}{3} - \frac{P_1 w l^4}{4} + \frac{w^2 l^5}{20}\right].$$

Now the condition for a minimum is $\frac{dV}{dR} = 0$, or, since P_1 is a function of R, we have

$$\frac{dV}{dR} = \frac{\partial V}{\partial P_1}\cdot\frac{dP_1}{dR} = \frac{1}{EI}\left[\frac{2P_1 l^3}{3} - \frac{wl^4}{4}\right]\left(-\frac{1}{2}\right) = 0,$$

whence $$P_1 = \tfrac{3}{8}wl, \text{ and } R = \tfrac{5}{4}wl.$$

390. Three 4 in. × 4 in. beams are placed 3 ft. apart on centers across an opening 20 ft. wide. Across the middle is placed another beam the same size, and upon the middle of this there rests a load of 1 T. (Fig. 217). Find the amount of the load carried by each beam.

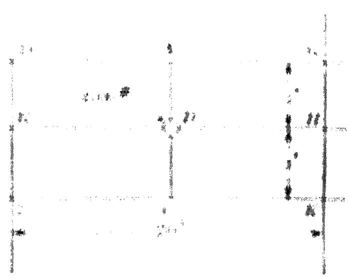


FIG. 217

SOLUTION. Let P denote the amount of the load carried by the cross beam AC. Then the loads on DG and FK at A and C respectively are each $\frac{P}{2}$, and the load carried by EH is $2000 - P$. Now for a simple beam of length l carrying a load W at its center, the moment at a point distant x from the left support is $T = \frac{Wx}{2}$, and hence the work of deformation is

$$V = \int_0^l \frac{T^2 dx}{2EI} = 2\int_0^{l/2} \frac{W^2x^2dx}{8EI} = \frac{W^2 l^3}{96EI}.$$

Therefore in the present case the work of deformation for the beams GD and KF is

$$V_{GD} = V_{KF} = \frac{P^2 l_1^3}{384\, EI},$$

where l_1 denotes the lengths of these beams. Similarly for HE we have

$$V_{HE} = \frac{(2000 - P)^2 l_1^3}{96\, EI},$$

and for AC

$$V_{AC} = \frac{P^2 l_2^3}{96\, EI}.$$

Therefore the total work of deformation for the entire construction is

$$V = \frac{P^2 l_1^3}{192\, EI} + \frac{(2000 - P)^2 l_1^3}{96\, EI} + \frac{P^2 l_2^3}{96\, EI}.$$

Applying the condition for a minimum, we have

$$\frac{dV}{dP} = 0 = \frac{P l_1^3}{96\, EI} - \frac{(2000 - P)\, l_1^3}{48\, EI} + \frac{P l_2^3}{48\, EI}.$$

Here E and I drop out, and inserting the given numerical values, namely, $l_1 = 240$ in., $l_2 = 72$ in., we find $P = 1309.8$ lb.

391. A beam 20 ft. long is supported at each end and at a point distant 5 ft. from the left end. It carries a load of 180 lb. at the left end, and 125 lb. at a point distant 6 ft. from the right end. Find the reactions of the supports (Fig. 248).

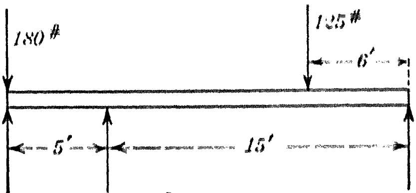


FIG. 248

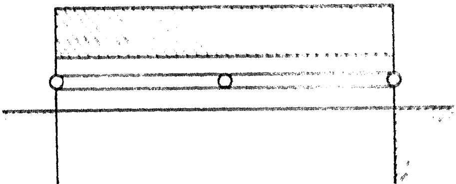


FIG. 249

392. Two beams are supported as shown in Fig. 249, the lower beam resting on fixed end supports, and the upper beam resting on three supports, at its center and ends. The upper beam carries a uniform load. Find the center load transmitted to the lower beam.

393. A flitched (or composite) beam consists of a 3-in. steel I beam and a 4 in. × 6 in. timber, and is hung from a crane hook by a strap around the center (Fig. 250). A cable is looped over the ends of the beam $2\frac{1}{2}$ ft. distant from the center on each side, and a load of 1000 lb. supported by it. Find the total load carried by each beam.

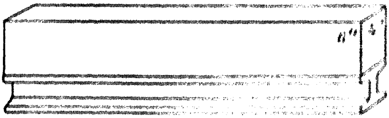


FIG. 250

(NOTE. — $I = 25$ in.^{4} for the steel I beam.)

394. The king post truss shown in Fig. 251 is formed of a single beam AC resting on supports at A and C and trussed at the center with a strut BD supported by two tie rods AD and DC. Determine the load R carried by the strut BD when a load P is placed at a distance c from A.

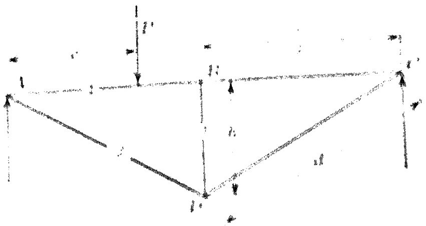


Fig. 251

SOLUTION. Let R denote the stress in BD. Then if h denotes the length of the strut BD and d the length of each tie AD and DC, the stress in AD or DC is $\frac{R}{2}\frac{d}{h}$ in each, and the direct stress in ABC is $\frac{R}{2}\cdot\frac{l}{2h}$. Let A_1, A_2, A_3 denote the cross sections of the members 1, 2, 3, as indicated in Fig. 251. Then the total work of deformation due to direct stresses in the various members is

$$V = \frac{R^2h}{2A_3E_3} + \frac{R^2d^3}{4h^2E_2A_2} + \frac{R^2l^3}{32E_1A_1h^2}.$$

In addition to this, it is also necessary to consider the work of deformation performed by the bending stress in AC. At a point distant x from A this is as follows:

$$\text{For } x \text{ between } A \text{ and } P, \quad Vx = \left[\frac{P(l-c)}{l} - \frac{R}{2}\right]x,$$

$$\text{For } x \text{ between } P \text{ and } B, \quad Vx = Pc - \left(\frac{R}{2} + \frac{Pc}{l}\right)x,$$

$$\text{For } x \text{ between } B \text{ and } C, \quad Vx = \left(\frac{Pc}{l} - \frac{R}{2}\right)(l - x).$$

Hence the total internal work due to bending is

$$V = \frac{1}{2E_1I_1}\left\{\int_0^c\left[\frac{P(l-c)}{l} - \frac{R}{2}\right]^2x^2dx + \int_c^{\frac{l}{2}}\left[Pc - \left(\frac{R}{2} + \frac{Pc}{l}\right)x\right]^2dx\right.$$

$$\left. + \int_{\frac{l}{2}}^{l}\left[\left(\frac{Pc}{l} - \frac{R}{2}\right)(l - x)\right]^2dx\right\}.$$

Now applying the condition $\frac{dV}{dR} = 0$ to the sum of these expressions, we obtain the equation

$$\frac{dV}{dR} = 0 = \frac{Rh}{E_3A_3} + \frac{Rd^3}{2E_2A_2h^2} + \frac{Rl^3}{16E_1A_1h^2} + \frac{1}{2E_1I_1}\left[-\frac{Pc^3(l-c)}{3l} + \frac{Rc^3}{6}\right.$$

$$\left. + \left\{\frac{R}{2} + \frac{P(l-c)}{l}\right\}\left(\frac{l^3 - 8c^3}{24}\right) - \frac{Pc(l^2 - 8c^2)}{24} - \frac{Pc(l^2 - 4c^2)}{8} - \frac{Pcl^2}{24} + \frac{Rl^3}{48}\right],$$

whence

$$R = \frac{\dfrac{3cl^2 - 4c^3}{48E_1I_1}}{\dfrac{h}{E_3A_3} + \dfrac{d^3}{2h^2E_2A_2} + \dfrac{l^3}{16h^2E_1A_1} + \dfrac{l^3}{48E_1I_1}}P.$$

395. A wooden beam 12 in. deep, 10 in. wide, and 20 ft. long between supports is reënforced by a steel rod 2 in. in diameter and a cast-iron strut 3 in. square and 2 ft. high, the whole forming a king post truss. Find the stress in each member due to a uniform load of 1200 lb./ft. over the entire beam.

396. The lattice truss, shown in Fig. 252, carries a load P at the middle panel point. Determine the stress in the various members.

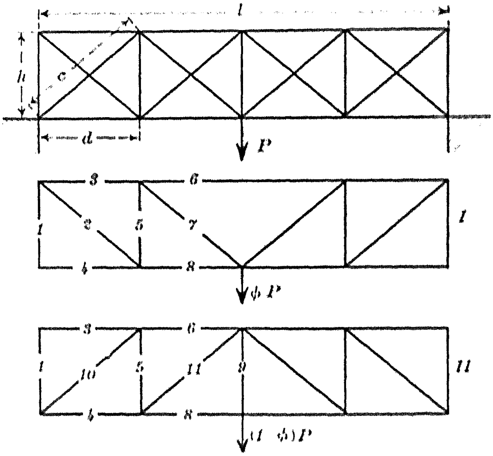


FIG. 252

SOLUTION. Since there are three members having vertical components which meet at the point where the load is applied, the statical conditions of equilibrium will not suffice to determine the stresses in these members. When any member of a truss can be removed without affecting the stability of the structure, such a member is called redundant. The truss here shown can evidently be separated into two parts, either of which forms a complete truss. The redundant members in this case are those which appear in one partial truss, but not in the other. Thus either the members numbered 9, 10, and 11 are redundant, or else the members 2 and 7.

To determine the distribution of stress in the various members by the principle of least work, consider the separate trusses numbered I and II and let ϕ denote the fraction of the load P carried by No. I. Then $(1 - \phi)P$ is the amount of the load carried by No. II. Since neither of these partial trusses contains any redundant members, the stresses in the various members can be found from the ordinary conditions of equilibrium. The results of this analysis for each truss is tabulated below.

I										
1	2	3	4	5	6	7	8	9	10	11
$\frac{1}{2}\phi P$	$\frac{c}{2h}\phi P$	$\frac{d}{2h}\phi P$	0	$\frac{1}{2}\phi P$	$\frac{d}{h}\phi P$	$\frac{c}{2h}\phi P$	$\frac{d}{2h}\phi P$			
II										
1	2	3	4	5	6	7	8	9	10	11
0		0	$\frac{d}{2h}(1-\phi)P$	$\frac{1}{2}(1-\phi)P$	$\frac{d}{2h}(1-\phi)P$		$\frac{d}{h}(1-\phi)P$	$(1-\phi)P$	$\frac{c}{2h}(1-\phi)P$	$\frac{c}{2h}(1-\phi)P$

Now forming the expression for the work of deformation due to these direct stresses in the various members, the total work of deformation for the entire truss is found to be as follows:

$$V = \sum \frac{P^2 l}{2EA} = \frac{\phi^2 P^2}{2E}\left\{\frac{h}{2}\left(\frac{1}{A_1}+\frac{1}{A_3}\right)+\frac{c^3}{2h^2}\left(\frac{1}{A_2}+\frac{1}{A_7}\right)+\frac{d^3}{2h^2}\left(\frac{1}{A_5}+\frac{1}{A_4}+\frac{1}{A_6}\right)\right\}$$

$$+\frac{(1-\phi)^2 P^2}{2E}\left\{\frac{h}{2}\left(\frac{1}{A_1}+\frac{2}{A_7}\right)+\frac{c^3}{2h^2}\left(\frac{1}{A_{10}}+\frac{1}{A_{11}}\right)+\frac{d^3}{2h^2}\left(\frac{1}{A_6}+\frac{1}{A_4}+\frac{1}{A_8}\right)\right\}.$$

Applying to this expression the condition for a minimum, namely, $\frac{dV}{d\phi} = 0$, and solving the resulting equation for ϕ, we have finally

$$\phi = \frac{\frac{h}{2}\left(\frac{1}{A_1}+\frac{2}{A_7}\right)+\frac{c^3}{2h^2}\left(\frac{1}{A_{10}}+\frac{1}{A_{11}}\right)+\frac{d^3}{2h^2}\left(\frac{1}{A_6}+\frac{1}{A_4}+\frac{1}{A_8}\right)}{\frac{h}{2}\left(\frac{1}{A_1}+\frac{2}{A_3}+\frac{2}{A_7}\right)+\frac{c^3}{2h^2}\left(\frac{1}{A_2}+\frac{1}{A_7}+\frac{1}{A_{10}}+\frac{1}{A_{11}}\right)+\frac{d^3}{2h^2}\left(\frac{1}{A_5}+\frac{4}{A_4}+\frac{4}{A_6}+\frac{1}{A_8}\right)}.$$

397. Determine the condition for the stability of an arch by means of the principle of least work.

SOLUTION. Consider a section of the arch perpendicular to the center line of the arch ring. Let A denote the area of this section, R the resultant pressure on it, e the distance of the point of application of R from the center of gravity of the section, and ds an infinitesimal element of the center line of the arch. Then the work of deformation will consist of two parts: that due to the axial thrust R, and that due to the moment $T = Re$. Since the direct stress per unit of area of the section is $\frac{R}{A}$, the unit deformation due to this stress is by Hooke's law equal to $\frac{R}{EA}$, where E denotes Young's modulus, or constant ratio of stress to deformation. Consequently the work of deformation due to R is, for a unit length, equal to $\frac{1}{2}R\left(\frac{R}{AE}\right)$, or $\frac{R^2}{2AE}$, and for an infinitesimal element of length ds is $\frac{R^2 ds}{2AE}$. Since the work of deformation due to the bending moment T is $\frac{T^2 ds}{2EI}$, the total work of deformation for the entire arch is

$$V = \int \frac{R^2 ds}{2AE} + \int \frac{T^2 ds}{2EI}.$$

Now in the present case R and T depend upon three unknowns which may conveniently be chosen as the position, amount, and direction of the pressure at any given point of the arch. Having so expressed R and T, the principle of least work is applied by differentiating V partially with respect to each of the three unknowns, and equating these three partial derivatives to zero. In this way three simultaneous equations are obtained which may be solved for the three unknown quantities, thus completely determining the pressure line, or linear arch, as it is called.

Instead of actually carrying out this process, it may be abbreviated into a simple criterion for stability. For this purpose let b denote the thickness of the arch ring, and consider a section of unit width. Then $A = b$ and $I = \frac{b^3}{12}$. Substituting these values in the above expression for V, it becomes

$$V = \frac{1}{2E}\int_0^l\left(\frac{R^2}{b} + \frac{12\,T^2}{b^3}\right)dx.$$

Now as the load on the arch changes, the pressure line alters its position, and consequently the value of this integral changes. The first term, however, changes but little in comparison with the second, since $T = Re$ and the distance e is the quantity which chiefly varies with the shifting of the pressure line. Hence the problem of making V a minimum reduces approximately to that of making $\int\frac{T^2}{b^3}dx$ as small as possible.

To effect a still further reduction, let R be resolved into vertical and horizontal components, R_V and R_H, so that the vertical component R_V shall pass through the center of gravity of the section (Fig. 253). Then $T = R_H z$, and the integral $\int\frac{T^2}{b^3}dx$ becomes $\int\frac{R_H^2 z^2}{b^3}dx$; or, since R_H is constant throughout the arch, this may be written $R_H^2\int\frac{z^2 dx}{b^3}$. Now ordinarily the thickness of an arch ring varies, being least at the crown and greatest at the abutments. Thus if b_c denotes the thickness at the crown, the thickness b at any other point may be assumed to be given approximately by the expression

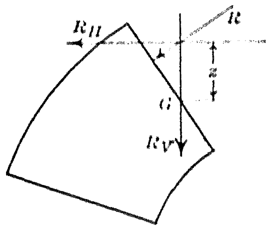


FIG. 253

$$b^3 = b_c^3\frac{dx}{dh},$$

where dh is the horizontal projection of the element of the center line dx. Under this assumption, the expression $R_H^2\int\frac{z^2 dx}{b^3}$ becomes $\frac{R_H^2}{b_c^3}\int z^2 dh$. Hence the problem of making V a minimum is now reduced to that of making $\int z^2 dh$ as small as possible. This latter expression, however, consists of only positive terms, and reduces to zero for the center line of the arch. From this it follows that if an equilibrium polygon is drawn for any given system of loads and the center line of the arch is then so chosen as to coincide with this equilibrium polygon, the actual pressure line or linear arch can differ but little from this center line. It can be shown, however, that for an arch to be stable the linear arch must fall within the middle third of the arch ring.*
Hence if for any given arch it is possible to draw an equilibrium polygon

* Slocum and Hancock, *Strength of Materials*, Revised Edition, p. 221.

which shall everywhere lie within the middle third of the arch ring, the stability of the arch is assured.

This is known as Winkler's criterion for stability, and was first given by him in 1879.

132. Hamilton's Principle. In Art. 128 it was shown that Lagrange deduced from d'Alembert's principle a set of dynamical equations by means of which the motion of a system possessing any number of degrees of freedom can be determined. A still more general dynamical principle was derived by Hamilton. This principle may be obtained either from Lagrange's equations or directly from d'Alembert's principle. To show the equivalence of Hamilton's principle with Lagrange's equations it will be deduced from the latter.

In the differential calculus it is shown that if in a function of one variable x, say $f(x)$, the variable is increased by an increment h, that

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2}+\cdots,$$

or, transposing,

$$f(x+h)-f(x)=f'(x)h+f''(x)\frac{h^2}{2}+\cdots.$$

Now if h is sufficiently small, the term in h^2 will be infinitesimal in comparison with that in h, and hence the change in the function, namely, $f(x+h)-f(x)$, will have the same sign as h. For a maximum or minimum value of the function, however, the change in the function must have the same sign whether h is positive or negative. The condition for this is that the term in h shall disappear, *i.e.* $f'(x)=0$, since in this case the sign will depend on the sign of $h^2f''(x)$, which is always the same as that of $f''(x)$ since h^2 is essentially positive.

Suppose now that instead of taking a variable point along a fixed curve that we change the form of the curve. Thus suppose that the point coordinates x, y, z are functions of an independent variable t such that by giving different values to t we get different points in space. Then if $f(x, y, z, x', y', z', \text{etc.})$ denotes a certain curve, by changing t the variables entering into this equation will all be changed, and consequently the given curve will be transformed into another curve of the same family. If t

changes continuously, the transformation will also be continuous; that is to say, the curve will move continuously from its original position into the new. Let δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$ ⋯ denote the changes, or variations, produced in the variables occurring in the equation of the curve by a given change in t. Then expanding the function by Taylor's theorem for several variables, we have

$$f(x+\delta x,\ y+\delta y,\ z+\delta z,\ x'+\delta x',\ y'+\delta y',\ \cdots)-f(x,\ y,\ z,\ x',\ y',\ \cdots)$$

$$=\left(\delta x\frac{\partial}{\partial x}+\delta y\frac{\partial}{\partial y}+\delta z\frac{\partial}{\partial z}+\delta x'\frac{\partial}{\partial x'}+\delta y'\frac{\partial}{\partial y'}+\cdots\right)f$$

$$+\frac{1}{\lfloor 2}\left(\delta x\frac{\partial}{\partial x}+\delta y\frac{\partial}{\partial y}+\delta z\frac{\partial}{\partial z}+\delta x'\frac{\partial}{\partial x'}+\delta y'\frac{\partial}{\partial y'}+\cdots\right)^2 f+\cdots.$$

Since the first term on the right contains only the first power of the δ's, while the second contains their squares and products, the change in the function, whether positive or negative, will depend on the sign of the first parenthesis. This term is called the first variation of the function and is denoted by δf; that is,

$$\delta f=\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial z}\delta z+\frac{\partial f}{\partial x'}\delta x'+\frac{\partial f}{\partial y'}\delta y'+\cdots.$$

This may be remembered by noting that it is identical in form with the ordinary calculus expression for a perfect differential.

Referring now to Lagrange's equations in Art. 128, we have

$$F_i=\frac{d}{dt}\left(\frac{\partial E}{\partial q_i'}\right)-\frac{\partial E}{\partial q_i},$$

where F_i denotes the generalized force, E is the kinetic energy of the system, and q_i, q_i' are generalized coördinates. The only case here considered is that for conservative systems, namely, those in which the force can be derived from a potential function V, or

$$F_i=-\frac{\partial V}{\partial q_i}.$$

Making this substitution for F_i in Lagrange's equations, they take the form

$$-\frac{\partial V}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial E}{\partial q_i'}\right)-\frac{\partial E}{\partial q_i},$$

or transposing, $$\frac{\partial(E-V)}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_k}\right).$$

Now multiply this equation through by δq_k and write a similar expression for each of the other coordinates. Then we have

$$\frac{\partial(E-V)}{\partial q_1}\delta q_1 = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right)\delta q_1,$$

$$\frac{\partial(E-V)}{\partial q_2}\delta q_2 = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_2}\right)\delta q_2,$$

etc.

Adding all these together, we obtain the relation

$$\frac{\partial(E-V)}{\partial q_1}\delta q_1 + \frac{\partial(E-V)}{\partial q_2}\delta q_2 + \cdots = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right)\delta q_1 + \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_2}\right)\delta q_2 + \cdots \quad (\mathrm{I})$$

Now the potential energy V is a function of the point coordinates $q_1, q_2 \cdots$ only, whereas the kinetic energy E is a function of both the coordinates $q_1, q_2 \cdots$ and the velocities $\dot{q}_1, \dot{q}_2 \cdots$. Hence referring to the expression for the variation of a function deduced above, and applying it to the function $E - V$, we obtain the identity

$$\delta(E-V) = \frac{\partial(E-V)}{\partial q_1}\delta q_1 + \frac{\partial(E-V)}{\partial q_2}\delta q_2 + \cdots + \frac{\partial E}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial E}{\partial \dot{q}_2}\delta \dot{q}_2 + \cdots \quad (\mathrm{II})$$

* The derivatives of V with respect to the $\dot{q}$'s do not appear in this expression because V is a function of the coordinates q only, and contains no $\dot{q}$'s.

Eliminating by subtraction the terms in δq_1, δq_2, etc., between equations (I) and (II), the result is

$$\delta(E-V) = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right)\delta q_1 + \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_2}\right)\delta q_2 + \cdots + \frac{\partial E}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial E}{\partial \dot{q}_2}\delta \dot{q}_2 + \cdots$$

Now, since $\frac{d}{dt}\left(\frac{\partial E}{\partial q_1'}\delta q_1\right)=\frac{d}{dt}\left(\frac{\partial E}{\partial q_1'}\right)\delta q_1+\frac{\partial E}{\partial q_1'}\frac{d}{dt}\delta q_1,$

and since the operations of variation and differentiation are commutative,* that is,

$$\frac{d}{dt}\delta q_1=\delta\frac{dq_1}{dt}=\delta q_1',$$

we have $\frac{d}{dt}\left(\frac{\partial E}{\partial q_1'}\delta q_1\right)=\frac{d}{dt}\left(\frac{\partial E}{\partial q_1'}\right)\delta q_1+\frac{\partial E}{\partial q_1'}\delta q_1'.$

Hence substituting this equality in the last expression for $\delta(E-V)$, it becomes

$$\delta(E-V)=\frac{d}{dt}\left[\frac{\partial E}{\partial q_1'}\delta q_1+\frac{\partial E}{\partial q_2'}\delta q_2+\cdots\right].$$

Now integrating this expression with respect to the independent variable t between any two fixed points, say those which correspond to the times t_0 and t_1, we have

$$\int_{t_0}^{t_1}\delta(E-V)\,dt=\left[\frac{\partial E}{\partial q_1'}\delta q_1+\frac{\partial E}{\partial q_2'}\delta q_2+\cdots\right]_{t_0}^{t_1},$$

or, since the total change in the integral is equal to the sum of the changes in its elements,† this may be written

$$\delta\int_{t_0}^{t_1}(E-V)\,dt=\left[\frac{\partial E}{\partial q_1'}\delta q_1+\frac{\partial E}{\partial q_2'}\delta q_2+\cdots\right]_{t_0}^{t_1}.$$

The only restriction that has been imposed on the displaced motion so far is that the difference between it and the actual motion shall be small. Now introduce the restriction that at certain times t_0 and t_1, the configuration in the displaced motion, shall be identical with that in the actual motion. Then at the times t_0 and t_1 each $\delta q=0$, and hence

$$\delta\int_{t_0}^{t_1}(E-V)\,dt=0,$$

or interchanging the operations of variation and integration,

$$\int_{t_0}^{t_1}\delta(E-V)\,dt=0.$$

* See footnote, p. 359. † See footnote, p. 359.

This equation is the symbolic expression of Hamilton's principle. Lagrange's equations of motion have thus been transformed into a single condition involving only the kinetic and potential energy of the system, and entirely independent of its mechanism. From this relation, therefore, the motion of all parts of the system can be determined as soon as its kinetic and potential energy are known, irrespective of the mechanism of its various parts.

The quantity $E - V$ is called the Lagrangian function and is usually denoted by L. Writing Hamilton's principle in the form

$$\delta\int_{t_0}^{t_1} L\,dt = 0,$$

it is evident that it expresses the condition that the integral of the Lagrangian function shall be a maximum or minimum for the path actually traversed. This interpretation of Hamilton's principle, however, has no especial bearing on its application.

133. Application of Hamilton's Principle to the Transverse Vibrations of Beams and Rods. To illustrate the practical application of Hamilton's principle, the problem of the transverse vibration of a beam or rod will now be solved by its use. Let the longitudinal axis of the beam be taken for the X axis, and the vertical axis, or direction of vibration, for the Y axis. Then the velocity of any point of the beam will be $\frac{\partial y}{\partial t}$, and the kinetic energy of a particle of mass dm will be $\frac{1}{2}dm\left(\frac{\partial y}{\partial t}\right)^2$. Let l denote the length of the rod or beam, A the area of its cross section, and ρ the density of the material. Then for a section of length dx we have $dm = \rho A dx$. Using this value for dm, in the expression $\frac{1}{2}dm\left(\frac{\partial y}{\partial t}\right)^2$, and summing up for the entire beam, the kinetic energy of the beam is found to be

$$E = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial y}{\partial t}\right)^2 dx.$$

In the theory of beams it is shown that the work of deformation, or potential energy, of a bent beam is given by the expression*

$$V = \frac{1}{2}\int_0^l \frac{M^2 dx}{EI},$$

* Slocum and Hancock, *Strength of Materials*, Revised Edition, p. 94

where $E =$ Young's modulus of elasticity for the material,

$I =$ the static moment of inertia of a cross section,

$B =$ external bending moment $= EI \dfrac{d^2y}{dx^2}$.

Substituting this value of the external moment B in the expression for V, and using the partial derivative, since y is a function of more than one variable, we have

$$V = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx.$$

Inserting these values of the kinetic and potential energy E and V in Hamilton's principle, namely,

$$\delta\int_{t_0}^{t_1}(E - V)\,dt = 0,$$

the resulting condition is

$$\delta\int_{t_0}^{t_1}\frac{dt}{2}\int_0^l\left[\rho A\left(\frac{\partial y}{\partial t}\right)^2 - EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2\right]dx = 0.$$

Now the first step is to perform the operation of variation, indicated by the δ, on the quantity under the integral sign. Since ρ and A are constants, the first term gives

$$\delta\left(\frac{\partial y}{\partial t}\right)^2 = 2\frac{\partial y}{\partial t}\,\delta\frac{\partial y}{\partial t} = 2\frac{\partial y}{\partial t}\frac{\partial \delta y}{\partial t}.$$

To simplify this operation on the second term, put $z = \dfrac{\partial y}{\partial x}$. Then $\dfrac{\partial z}{\partial x} = \dfrac{\partial^2 y}{\partial x^2}$, and consequently

$$\delta\left(\frac{\partial^2 y}{\partial x^2}\right)^2 = \delta\left(\frac{\partial z}{\partial x}\right)^2 = 2\frac{\partial z}{\partial x}\frac{\partial \delta z}{\partial x}.$$

Substituting these values in the above condition, it becomes

$$\int_{t_0}^{t_1}dt\int_0^l\left[\rho A\frac{\partial y}{\partial t}\frac{\partial \delta y}{\partial t} - EI\frac{\partial z}{\partial x}\frac{\partial \delta z}{\partial x}\right]dx = 0.$$

Now integrating the first term of this expression partially with respect to t, we have

$$\int_{t_0}^{t_1}\rho A\frac{\partial y}{\partial t}\frac{\partial \delta y}{\partial t}dt = \rho A\frac{\partial y}{\partial t}\,\delta y\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\rho A\frac{\partial^2 y}{\partial t^2}\,\delta y\,dt.$$

Since δy vanishes for both limits t_1 and t_0, the integrated part of this expression is zero. Similarly by integrating the second term partially with respect to x, we have

$$\int_0^l EI \frac{\partial z}{\partial x} \frac{\partial \delta z}{\partial x} dx = \left[EI \frac{\partial z}{\partial x} \delta z \right]_0^l - \int_0^l EI \frac{\partial^2 z}{\partial x^2} \delta z dx.$$

Since both ends of the beam are assumed to be stationary, δy vanishes at the ends. Therefore, since $\delta z = \dfrac{\partial \delta y}{\partial x}$, δz also vanishes at both limits 0 and l. Consequently the integrated part also vanishes in this case.

Now substituting these transformed integrals in the double integral, the latter becomes

$$\int_{t_0}^{t_1} dt \int_0^l \left(-\rho A \frac{\partial^2 y}{\partial t^2} \delta y + EI \frac{\partial^2 z}{\partial x^2} \delta z \right) dx = 0,$$

or replacing z by its value $\dfrac{\partial y}{\partial x}$, this becomes

$$\int_{t_0}^{t_1} dt \int_0^l \left(-\rho A \frac{\partial^2 y}{\partial t^2} \delta y + EI \frac{\partial^3 y}{\partial x^3} \frac{\partial \delta y}{\partial x} \right) dx = 0.$$

Here again the second term must be transformed in order to obtain δy explicitly. Integrating by parts as before, we have

$$\int_0^l EI \frac{\partial^3 y}{\partial x^3} \frac{\partial \delta y}{\partial x} dx = \left[EI \frac{\partial^3 y}{\partial x^3} \delta y \right]_0^l - \int_0^l EI \frac{\partial^4 y}{\partial x^4} \delta y dx.$$

Since δy is zero at both limits, the integrated part vanishes. Hence substituting the transformed expression in the double integral, the latter finally becomes

$$\int_{t_0}^{t_1} dt \int_0^l \left(-\rho A \frac{\partial^2 y}{\partial t^2} - EI \frac{\partial^4 y}{\partial x^4} \right) \delta y dx = 0.$$

Since this relation must hold for all values of δy, i.e. it must be independent of δy, the coefficient of δy must be zero. Consequently $-\rho A \dfrac{\partial^2 y}{\partial t^2} - EI \dfrac{\partial^4 y}{\partial x^4} = 0$, or

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EI}{\rho A} \frac{\partial^4 y}{\partial x^4}.$$

This is the required differential equation of motion. The problem is thereby solved so far as the application of Hamilton's principle is concerned.

The simplest method of solving this differential equation is to find a particular solution of as simple a form as possible. This solution will then represent a possible form of vibration, and by a combination of such solutions the general solution representing a complex vibration may be obtained. In the present case the simplest solution of the above equation is of the general form

$$y = K \sin \alpha x \cos \beta t,$$

where K is an arbitrary constant representing the maximum amplitude, and α and β are to be determined from the conditions of the problem. Substituting this solution in the above differential equation and cancelling common factors, we find as the necessary relation between α and β,

$$EI\alpha^4 = \rho A \beta^2.$$

Since $y = 0$ at both ends of the beam, *i.e.* for $x = 0$ and $x = l$, and the vibration is independent of the time, *i.e.* $\cos \beta t \neq 0$, we also have the conditions that $\sin \alpha \cdot 0 = 0$ and $\sin \alpha l = 0$. The first of these conditions is merely an identity, but the second gives $\alpha l = n\pi$, where n is an arbitrary integer. If $n = 1$, the beam is said to have its fundamental vibration, and if it is of such dimensions as to give out a musical note, this is said to be its fundamental tone. In this case

$$\alpha = \frac{\pi}{l}, \quad \beta = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}},$$

and consequently the solution becomes

$$y = K \sin \frac{\pi x}{l} \sin \frac{\pi^2 t}{l^2}\sqrt{\frac{EI}{\rho A}}.$$

During a complete oscillation the angle βt increases by 2π. Hence the period of the oscillation is

$$P = \frac{2\pi}{\beta} = \frac{2}{\pi}\frac{l^2}{1}\sqrt{\frac{\rho A}{EI}}.$$

If n has any other value than unity, nodes occur. That is to say, the beam or rod is divided up into n equal parts, and at each of the points of division no motion occurs. The beam thus vibrates in segments, and the notes produced are called overtones to distinguish them from the fundamental tone. Under the same external conditions, these overtones are higher in pitch and less in amplitude, *i.e.* weaker, than the fundamental tone.

PROBLEMS

398. Obtain the differential equation for the vibration of an elastic string of length l fastened at the ends (Fig. 254).

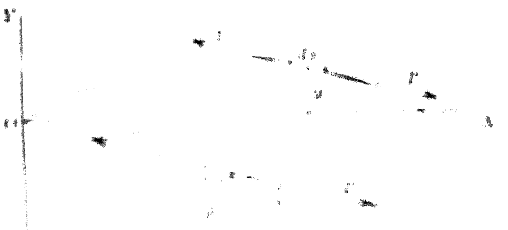


FIG. 254

SOLUTION. The kinetic energy of vibration is in this case

$$T = \frac{1}{2}\int_0^l \rho\left(\frac{\partial y}{\partial t}\right)^2 dx,$$

where dx denotes an element of length of the string.

Let P denote the tension in the string at any point. Then the vertical (or transverse) component of the tension is $P\frac{\partial y}{\partial x}$, and the distance through which this force acts is $dx\frac{\partial y}{\partial x}$. Hence the potential energy stored up in the string at any instant is

$$\frac{1}{2}\int_0^l P\left(\frac{\partial y}{\partial x}\right)^2 dx.$$

Substituting these values in Hamilton's principle, the resulting condition is

$$\frac{1}{2}\int_{t_0}^{t_1} dt\,\delta\int_0^l\left[\rho\left(\frac{\partial y}{\partial t}\right)^2 - P\left(\frac{\partial y}{\partial x}\right)^2\right]dx = 0.$$

Performing the operation of variation on this integral, the condition finally reduces to

$$\int_{t_0}^{t_1} dt\int_0^l\left(\rho\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2}\right)\delta y\,dx = 0.$$

Therefore since the variation is independent of δy, if the constant factor $\frac{P}{\rho}$ is denoted by a^2, the differential equation of motion becomes

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}.*$$

* For the solution of this differential equation and general discussion of its physical meaning, see Byerly, *Fourier's Series and Spherical Harmonics*, p. 7.

399. A cantilever beam projects from a wall, and a weight W falls on it. The beam breaks at a point on the opposite side of the weight from the wall. Show how to determine where the break will occur.

The solution of this problem is obtained from the differential equation of the preceding article, namely, $\frac{\partial^2 y}{\partial t^2} = -\frac{EI}{\rho A}\frac{\partial^4 y}{\partial x^4}$, by integrating it separately for the portions of the beam on opposite sides of the point where the weight strikes it, and determining the constants of integration by the terminal conditions. This problem is fully treated by St. Venant in the French translation of Clebsch on the Theory of Elasticity.

A similar problem is involved in the experiment of shooting a bullet through a pane of glass without breaking the glass except for the hole in which the bullet pierced it. This phenomenon is due to the fact that in the neighborhood of the hole very great inertia forces are created, which at first equilibrate the pressure between bullet and glass. These, however, extend over a very limited region, and beyond this are not sufficiently great to rupture the glass.

134. Theory of Models. — Experimental work in engineering is frequently carried out on a small model of the machine or structure to be tested. The object in using a model is to avoid the expense of a full-sized construction, or in certain cases to reduce the size so that tests can be made in an ordinary testing machine without special apparatus.

The model for such small scale testing is first constructed geometrically similar in all its parts to the given construction. Thus suppose that two machines are constructed from the same working drawings, the given dimensions in one case being read in feet and in the other case in inches. The two constructions will then be geometrically similar, any two corresponding dimensions being in the constant ratio of 1 : 12. From the fact that the two machines are geometrically similar, however, it does not follow that they will be dynamically similar, for experience has shown that one machine may run with entire satisfaction, while the other may be equally unsatisfactory. Tests carried out on a model, or miniature, are therefore unreliable unless care is taken to secure dynamical as well as geometrical similarity.

As a simple illustration, suppose that a model of a steam engine is constructed, geometrically similar to the original. Let λ denote the ratio in which the scale is reduced, so that any given dimension of the original is λ times the corresponding dimension in the model. The area of the piston is then reduced in the ratio

of $1:\lambda^2$, and hence if the steam pressure is the same in both cases, the accelerating force acting on the piston is also reduced in this ratio. The masses of the moving parts, however, are proportional to their volumes, and are therefore reduced in the model in the ratio of $1:\lambda^3$. Hence the acceleration produced in the model will not correspond to that of the original, and the two machines will therefore behave differently. The two can be made dynamically similar, however, by increasing the steam pressure in the smaller to λ times its value in the original. The ratio of pressure to mass will then be the same in both engines, and consequently the same acceleration will be produced in both.

As another simple illustration consider a small model of a loaded beam, and, as before, suppose each dimension diminished in the ratio of $1:\lambda$. If the material of the model is the same as that of the original, the elastic constants will be the same for both. The fundamental formula for stress in beams, however, is

$$p = \frac{Be}{I},$$

where p = unit fiber stress in lb./in.^{2},
B = external bending moment in in. lb.,
e = distance of extreme fiber from neutral axis in in.,
I = static moment of inertia of a cross section with respect to the neutral axis in in.^{4}.

Let the quantities in this formula refer to the model. Then, in the original beam, e and I will become $e' = \lambda e$ and $I' = \lambda^4 I$. Hence, if the unit fiber stress p is to be the same in both, we must have $p = \frac{Be}{I} = \frac{B'e'}{I'}$, and consequently the external moment for the full-sized beam must be $B' = \lambda^3 B$. Or, since the moment B is the product of a force P and a lever arm d, say $B = Pd$, and the lever arm for the full-sized beam is λd, the force must be $\lambda^2 P$. Consequently, in order for the beams to be dynamically similar, the loads must be in the ratio of $1:\lambda^2$.

The preceding considerations simply take account of the external loads on the beams, and neglect their own weights. If the weight of each beam is also to be taken into account, then, since the weight is proportional to the volume, the weights of the model

and the full-sized beam will be in the ratio of $1 : \lambda^3$. Hence, the full-sized beam is more severely strained by its own weight than the model. To make them dynamically similar in this respect, it is therefore necessary to distribute over the model an additional load equal to $(\lambda - 1)$ times its own weight. The ratio of the uniform loads will then be $\lambda : \lambda^3$ or $1 : \lambda^2$ as above.

For example: suppose it is desired to determine the deflection and other properties of a rectangular steel beam 20 ft. long and 8 in. square, by means of a quarter-sized model. Here $\lambda = 4$, and hence the dimensions of the model are 5 ft. long by 2 in. square. The weights of the two beams are then in the ratio of $1 : \lambda^3$ or $1 : 64$. Thus, assuming that steel weighs 490 lb./ft.^{3}, the weight of the full-sized beam is $4355\frac{5}{9}$ lb. and that of the model is $68\frac{1}{18}$ lb. It is therefore necessary to increase the uniform load carried by the model to $272\frac{2}{9}$ lb., by adding a distributed load of amount equal to $272\frac{2}{9} - 68\frac{1}{18}$, or $204\frac{1}{6}$ lb. When this load is added, the two beams will be dynamically similar, since the ratio of the loads, including their own weights, is now $272\frac{2}{9} : 4355\frac{5}{9} = 1 : 16 = 1 : \lambda^2$.

For columns, the same relation holds; namely, that for dynamical similarity the load on the full-sized column must be λ^2 times that on the model. This is evident from Euler's column formula $P = \dfrac{\pi^2 EI}{l^2}$. Here π^2 and E are constants. The moment of inertia I, however, is increased in the full-sized column in the ratio of $1 : \lambda^4$, and l^2 in the ratio of $1 : \lambda^2$. Consequently, P must also be increased in the ratio of $1 : \lambda^2$ if the formula is to be homogeneous in its dimensions.

The ideas illustrated by the preceding examples will now be amplified into a general theory of models. In mechanics four fundamental quantities are involved: length, time, mass, and force. These four quantities, however, are not all independent, since Newton's law of motion as expressed by the formula $F = ma$ establishes a relation between them. Let the ratios of these four quantities in the model and original be denoted as follows:

$\lambda =$ ratio of corresponding linear dimensions,
$\mu =$ ratio of corresponding masses,
$\tau =$ ratio of corresponding times,
$\pi =$ ratio of corresponding forces.

of $1 : \lambda^2$, and hence if the steam pressure is the same in both cases, the accelerating force acting on the piston is also reduced in this ratio. The masses of the moving parts, however, are proportional to their volumes, and are therefore reduced in the model in the ratio of $1 : \lambda^3$. Hence the acceleration produced in the model will not correspond to that of the original, and the two machines will therefore behave differently. The two can be made dynamically similar, however, by increasing the steam pressure in the smaller to λ times its value in the original. The ratio of pressure to mass will then be the same in both engines, and consequently the same acceleration will be produced in both.

As another simple illustration consider a small model of a loaded beam, and, as before, suppose each dimension diminished in the ratio of $1 : \lambda$. If the material of the model is the same as that of the original, the elastic constants will be the same for both. The fundamental formula for stress in beams, however, is

$$p = \frac{Bc}{I},$$

where p = unit fiber stress in lb./in.^{2},
B = external bending moment in in.-lb.,
c = distance of extreme fiber from neutral axis in in.,
I = static moment of inertia of a cross section with respect to the neutral axis in in.^{4}

Let the quantities in this formula refer to the model. Then, in the original beam, c and I will become $c' = \lambda c$ and $I' = \lambda^4 I$. Hence, if the unit fiber stress p is to be the same in both, we must have $p = \frac{Bc}{I} = \frac{B'c'}{I'}$, and consequently the external moment for the full-sized beam must be $B' = \lambda^3 B$. Or, since the moment B is the product of a force P and a lever arm d, as $B = Pd$, and the lever arm for the full-sized beam is λd, the force must be $\lambda^2 P$. Consequently, in order for the beams to be dynamically similar, the loads must be in the ratio of $1 : \lambda^2$.

The preceding considerations simply take account of the external loads on the beams, and neglect their own weights. If the weight of each beam is also to be taken into account, then, since the weight is proportional to the volume, the weights of the model

and the full-sized beam will be in the ratio of $1 : \lambda^3$. Hence, the full-sized beam is more severely strained by its own weight than the model. To make them dynamically similar in this respect, it is therefore necessary to distribute over the model an additional load equal to $(\lambda - 1)$ times its own weight. The ratio of the uniform loads will then be $\lambda : \lambda^3$ or $1 : \lambda^2$ as above.

For example: suppose it is desired to determine the deflection and other properties of a rectangular steel beam 20 ft. long and 8 in. square, by means of a quarter-sized model. Here $\lambda = 4$, and hence the dimensions of the model are 5 ft. long by 2 in. square. The weights of the two beams are then in the ratio of $1 : \lambda^3$ or $1 : 64$. Thus, assuming that steel weighs 490 lb./ft.^{3}, the weight of the full-sized beam is $4355\frac{5}{9}$ lb. and that of the model is $68\frac{1}{18}$ lb. It is therefore necessary to increase the uniform load carried by the model to $272\frac{2}{9}$ lb., by adding a distributed load of amount equal to $272\frac{2}{9} - 68\frac{1}{18}$, or $204\frac{1}{6}$ lb. When this load is added, the two beams will be dynamically similar, since the ratio of the loads, including their own weights, is now $272\frac{2}{9} : 4355\frac{5}{9} = 1 : 16 = 1 : \lambda^2$.

For columns, the same relation holds; namely, that for dynamical similarity the load on the full-sized column must be λ^2 times that on the model. This is evident from Euler's column formula $P = \dfrac{\pi^2 EI}{l^2}$. Here π^2 and E are constants. The moment of inertia I, however, is increased in the full-sized column in the ratio of $1 : \lambda^4$, and l^2 in the ratio of $1 : \lambda^2$. Consequently, P must also be increased in the ratio of $1 : \lambda^2$ if the formula is to be homogeneous in its dimensions.

The ideas illustrated by the preceding examples will now be amplified into a general theory of models. In mechanics four fundamental quantities are involved: length, time, mass, and force. These four quantities, however, are not all independent, since Newton's law of motion as expressed by the formula $F = ma$ establishes a relation between them. Let the ratios of these four quantities in the model and original be denoted as follows:

λ = ratio of corresponding linear dimensions,
μ = ratio of corresponding masses,
τ = ratio of corresponding times,
π = ratio of corresponding forces.

Now the dimensional equation resulting from Newton's law $F = ma$ is

$$F = \frac{ML}{T^2}.$$

Hence, for dynamical similarity between model and original the above ratios must satisfy a similar relation, namely, $\pi = \frac{\mu\lambda}{\tau^2}$, or

$$\text{(I)} \qquad \pi\tau^2 = \mu\lambda.$$

If the weights of the various parts are to be taken into consideration as well as the external forces, then, since the weight is proportional to the mass, and in order to produce equal accelerations, the forces must also be proportional to the masses, we will have $\pi = \mu$. The above condition is therefore resolved into two conditions,

$$\text{(II)} \qquad \pi = \mu, \quad \tau^2 = \lambda.$$

Again, if the material is to be subject to the same unit stress in both cases, then, as in the example of the beam and its model, considered above, we must have $\pi = \lambda^2$, or, combining this with the preceding condition, we now have

$$\text{(III)} \qquad \pi = \lambda^2 = \mu = \tau^4.$$

These conditions, however, can only be fulfilled when the weight of the structure is neglected in comparison with the external forces. If the construction's own weight is to be considered, then we must have $\mu = \lambda^3$, and hence the last set of conditions is replaced by the following set:

$$\text{(IV)} \qquad \pi = \lambda^3 = \mu = \tau^6.$$

To illustrate the application of these conditions consider the model of a ship, and let it be required to determine the relative speed of the ship and its model for dynamical similarity, and also their relative resistance to motion at these speeds. Suppose that the ship is required to maintain a speed of 30 knots, and that it is desired to find what resistance it will offer at this speed by experiments with a model constructed to $\frac{1}{25}$ the scale of the ship. In this case $\lambda = 25$. Since the displacement of the ship, *i.e.* its weight, is an important factor, condition (IV) must be applied. Hence,

$\lambda^3 = \tau^6$, or $\lambda = \tau^2$, from which $\tau = 5$. Hence, for dynamical similarity the time occupied by the model in passing over a given distance must be 5 times that taken by the ship to pass over the corresponding distance, or, in other words, the speed of the model should be $\frac{1}{5}(30) = 6$ knots. To determine their relative resistance to motion, let R denote the resistance of the model as determined by experiment. Then the resistance of the ship will be πR, or since $\pi = \lambda^3 = 25^3$, the resistance of the ship will be $15{,}625\ R$.

As another example of the application of these conditions, suppose it is desired to determine the deflection of a bridge under moving loads by experiments on a small model. To make the problem definite, suppose that the bridge weighs 200 T. and is 50 ft. long, and that the moving load is due to a locomotive weighing 60 T., with a speed of 30 ft./sec. Assume the model of the bridge to be constructed to $\frac{1}{10}$ the scale, so that $\lambda = 10$. The relative weight of the model and the full-sized bridge will then be as $1 : 10^3$; that is, if the two are geometrically similar throughout, the weight of the model will be $\frac{200}{1000} = \frac{1}{5}$ T. = 400 lb., and it will be 5 ft. long. If, however, the material in each case is to be equally strained, condition (III) must be applied, *i.e.* the forces must be proportional to the cross section, which makes $\pi = \lambda^2$. To be equally strained by its own weight, the model should therefore weigh $\frac{200}{\lambda^2} = 2$ T. Since its actual weight is only 400 lb., the difference of 3600 lb. may be made up by hanging this amount at the various joints of the truss, in which case the two constructions will be dynamically similar.

If, however, it is certain that the material of the bridge is not strained beyond the elastic limit, the difference in dead load due to its greater proportional weight may be neglected, and the moving load on the model and its speed be so determined that mechanical similarity will still exist. Thus for the ratio of the weights (or masses) we have in the present case

$$\mu = \frac{400000}{400} = 1000 = \lambda^3 = \pi = \tau^6,$$

and $\tau = \sqrt{\lambda} = \sqrt{10} = 3.16$. The speed of the moving load on the model must therefore be $\frac{30}{3.16} = 9.5$ ft./sec., and its weight must

be $\frac{60}{\mu} = \frac{60}{1000} = 0.06$ T. $= 120$ lb. If, then, the deflection of the model at any point is D, that of the bridge at the corresponding point will be $\lambda D = 10\,D$.

135. Analogy between the Equations of Dynamics and Electrodynamics. The credit is due to Maxwell of treating a system of currents and the magnetic field belonging to them as a mechanical system subject to the ordinary laws of motion, and in this way deriving the equations of induction from the generalized equations of Lagrange and Hamilton.

The phenomenon of the induction of electric currents by changes in the magnetic field was discovered by Faraday in 1831, and deduced mathematically by F. E. Neumann in 1845. To give a physical explanation of inductance, Sir Oliver Lodge imagined the magnetic field surrounding a conductor to consist of whirls in the ether, driven by the current and possessing inertia somewhat like small flywheels. Any change in the electromotive force tends to produce a change in the current, which is opposed by the inertia of the whirls. It is found that this opposing action of the whirls, or magnetic field, is stronger the greater the flux, or current, and the more quickly the changes in electromotive force occur, that is, the greater their frequency, being analogous to the action of a flywheel driven at a non-uniform speed. This reaction of conductors in a magnetic field expressed numerically is called their **inductance**, the reason for the name being that the reaction consists of inducing a counter-electromotive force in the conductor. Inductance is thus the analogue of inertia.

Since the effect of moving a closed conductor in a magnetic field is to induce a current in the circuit, the magnetic field may be regarded as the source of kinetic energy. The amount of energy in any given portion of the field is proportional to the square of the intensity of the field, that is, to the square of the current. If, then, L denotes the inductance of the circuit, and i the current flowing through it, the kinetic energy of the magnetic field may be written

$$K.E. = \tfrac{1}{2} Li^2.$$

From what precedes, the inductance L is the analogue of inertia m or moment of inertia I in any mechanical system. The current

i is similarly analogous to the linear velocity v, or angular velocity ω, and the expression for electrodynamic energy corresponds precisely to the ordinary relations,

$$\tfrac{1}{2}\,mv^2, \text{ or } \tfrac{1}{2}\,I\omega^2.$$

One of the fundamental principles of induction is that the effect of an electromotive force e is to cause a change in the current in accordance with the relation

$$e = L\frac{di}{dt}.$$

This relation is analogous to the ordinary mechanical equations,

$$F = m\frac{dv}{dt}, \text{ or } T = I\frac{d\omega}{dt},$$

and consequently electromotive force corresponds to mechanical force or torque.

The phenomenon of resistance may be compared to that of molecular friction. Thus if R denotes the resistance of a circuit, the relation between the electromotive force and the current in the circuit is given by Ohm's law

$$e = Ri.$$

The electromotive force e is here analogous to the head or pressure in the case of water flowing through a pipe, in which case the greater the frictional resistance, the less will be the current.

Many other mechanical analogies may be deduced. Thus the product of force and velocity is mechanical power; for example, the product of the difference in tension in a belt in pounds by its speed in ft./min. divided by the constant 33,000 gives the horsepower it is transmitting. Similarly the product of electromotive force in volts and current in amperes gives the electric power in watts being generated.

As a further example, the action of a condenser in storing electric energy is similar to the ordinary phenomenon of elasticity. The electrostatic capacity of a condenser is thus analogous to the work of deformation of an elastic solid up to the elastic limit. This similarity is further apparent in the analogy between the phenomena of mechanical and electrical resonance. Electrical

resonance arises when inductance and capacity are both present in an A. C. circuit. Any given combination of inductance and capacity gives rise to electrical oscillations of a definite frequency. If the frequency of the supply happens to be a multiple or sub-multiple of this natural period, the amplitude of these oscillations is greatly intensified. This phenomenon is different according as the inductance and capacity are connected in parallel or in series. When connected in parallel, current resonance is produced, and when connected in series, voltage resonance results.

It is also a well-known principle that if the strength of a system of currents is maintained constant, the currents tend to move in such a way as to increase the energy of the field produced by them. This energy is a homogeneous quadratic function of the strengths of the various currents, of the form $\frac{1}{2} Li^2$, the coefficients L, called inductances, being determined by the form and position of the circuits, and the nature of the field in which they are situated. The medium being specified, these geometrical specifications of the circuits may be made by giving a certain finite or infinite number of geometrical parameters q_s, which correspond to Lagrange's generalized coordinates. The electromagnetic forces due to the action of the currents may be equilibrated by impressed forces P_s such that

$$P_s = \frac{\partial V}{\partial q_s},$$

where V denotes the potential energy of the field. In order to completely specify the action of the system, the values of the current strengths i_s must also be given. Thus for n linear conductors, the electrokinetic energy is given by the expression

$$E = \frac{1}{2} L_1 i_1^2 + M_{12} i_1 i_2 + \cdots + M_{1n} i_1 i_n + \frac{1}{2} L_2 i_2^2 + M_{23} i_2 i_3 + \cdots$$
$$+ M_{2n} i_2 i_n + \cdots + \frac{1}{2} L_n i_n^2,$$

where the L's denote the self-inductances of the various circuits, and the M's are the mutual inductances of the circuits indicated by the subscripts. The electrokinetic momentum p_s of any circuit is then given by

$$p_s = \frac{\partial E}{\partial i_s} = M_{1s} i_1 + \cdots + L_s i_s + \cdots + M_{ns} i_n.$$

and the corresponding force is $P_s = \frac{dp_s}{dt}$. (Compare the ordinary relations $E = \frac{1}{2}mv^2$, momentum $= \frac{\partial E}{\partial v} = mv$, and force $= \frac{d(mv)}{dt} = ma$.) This force, however, consists of the impressed electromotive force e_s, due to thermal, chemical, or other action, and the dissipative term $-R_s i_s$, where R_s denotes the resistance of the circuit. Hence

$$P_s = e_s - R_s i_s = \frac{dp_s}{dt},$$

which may be written

$$i_s = \frac{e_s - \frac{dp_s}{dt}}{R_s}.$$

Hence the current in any circuit may be calculated by Ohm's law by including the electromotive force of induction $-\frac{dp_s}{dt}$. If the current does not vary with the time and there is no motion of any circuit, then every p_s is constant and the above reduces to

$$i_s = \frac{e_s}{R_s},$$

which is the usual form of Ohm's law. The above is then the general equation for an electric current, including the steady state as a particular case as just shown, and the method by which it is obtained is precisely analogous to that by which Lagrange's generalized equations of motion were deduced.

CHAPTER VII

DYNAMICS OF ROTATION

136. Size and Weight of Flywheels.—In Chapter I, Art. 28, it was shown that the motion of the piston or cross head of a steam engine is approximately harmonic. That is to say, if the speed of the crank pin is assumed to be constant, the speed of the piston is represented approximately by the ordinates to a semicircle. Since the speed of the reciprocating parts is zero at either end of the stroke and a maximum at the middle, these parts are accelerated during the first half of the stroke and retarded during the second half. Consequently the energy spent in accelerating these parts during one half of the stroke is given up again during the other half, and unless equalized in some way, this alternate give and take of energy greatly affects the smooth running of the engine.

To diminish the effect of this inertia thrust and equalize the torque on the crank shaft, flywheels are ordinarily provided. Since a flywheel is capable of storing up energy on a much larger scale than the reciprocating parts alone, the fluctuation in speed during the first and the last halves of the stroke is much reduced.

Let

ω_1 = maximum angular velocity of flywheel or shaft,

ω_2 = minimum angular velocity of same,

ω = average angular velocity of same.

Then the total change in speed is $\omega_1 - \omega_2$, and the fluctuation is defined as the ratio of this change to the average angular velocity ω; that is,

$$\frac{\omega_1 - \omega_2}{\omega} = f = \text{fluctuation coefficient.}$$

Now let I denote the polar moment of inertia of the flywheel about its axis. Then

$$\text{Energy stored at speed } \omega_1 = \tfrac{1}{2} I\omega_1^2,$$

$$\text{Energy stored at speed } \omega_2 = \tfrac{1}{2} I\omega_2^2$$

The difference between these two is the excess energy put in by the steam, say E. That is,

$$\text{Excess energy} = E = \tfrac{1}{2} I(\omega_1^2 - \omega_2^2)$$
$$= \tfrac{1}{2} I(\omega_1 + \omega_2)(\omega_1 - \omega_2).$$

Since $\omega_1 + \omega_2 = 2\,\omega$, and $\omega_1 - \omega_2 = \omega f$, this may be written

$$E = I\omega^2 f.$$

Now let $W =$ weight of flywheel in pounds,
$k =$ its radius of gyration in feet,
$N =$ number of revolutions per minute.

Then $I = \dfrac{Wk^2}{g}$, and $\omega = \dfrac{2\,\pi N}{60}$; consequently

$$E = \frac{Wk^2}{g} \cdot \frac{4\,\pi^2 N^2 f}{3600} = \frac{k^2 N^2 f W}{2937}.$$

Now if i. h. p. denotes the number of indicated horsepower of the engine, the work done by the engine is 33,000 i. h. p. ft.-lb./min. Consequently the work done per revolution is $\dfrac{33000 \text{ i. h. p.}}{N}$, and the work done per stroke is $\dfrac{33000 \text{ i. h. p.}}{2\,N}$. The excess energy stored in the flywheel per stroke is some fraction of this amount, say m. Then

$$E = \frac{m\, 33000 \text{ i. h. p.}}{2\,N},$$

Equating this value of E to that obtained above, we have

$$\frac{k^2 N^2 f W}{2937} = \frac{m\, 33000 \text{ i. h. p.}}{2\,N}$$

whence

$$W = \frac{2937\, m\, 33000 \text{ i. h. p.}}{2\, k^2 N^3 f} = \left(\frac{100}{N}\right)^3 \left(\frac{48.5\, m}{f k^2}\right) \text{i. h. p,}$$

which gives the required weight of flywheel for an engine of given i. h. p.

Since 1 h. p. = 0.746 kw., this becomes in terms of kilowatts

$$W = \left(\frac{100}{N}\right)^3 \left(\frac{65\, m}{f k^2}\right) \text{kw.}$$

137. Values of the Constants. For a well-designed flywheel, the radius of gyration, k, will be found to lie between 0.37 and 0.43 of the diameter. A good average value of k is obtained by assuming

$$k = 0.4\,D,$$

where D denotes the outside diameter of the flywheel.

The fluctuation coefficient, f, varies from $\frac{1}{20}$ to $\frac{1}{400}$ according to the class of service for which the engine is designed. The following are average values of f found in practice:

Electric lighting, A. C. in parallel	$f = 0.0025$
Ordinary electric lighting, A. C. or D. C.	0.005
Dynamos	0.007
Spinning machinery	0.02
Machine tools	0.03
Flour mills	0.04
Shearing and punching machines, pumps, etc.	0.05

The value of m, or excess energy E stored per stroke, depends on the style of engine, number of cylinders, point of cut-off, etc.

For steam engines, average values of m are

Single cylinder steam engine	$m = \frac{1}{4}$
Two cylinders, cross compound	$\frac{1}{12}$
Three cylinders, cranks 120° apart	$\frac{1}{27}$
Gas engines exploding every 4th stroke	4.5
" " " " 8th "	8.5

It should be noted, however, that the value of m may be 100 per cent in excess of the average values here given. For n cylinders the general rule is that

$$m = \frac{1}{n^2} \times \text{value of } m \text{ for one cylinder.}$$

For large flywheels, such as those used on Corliss engines, it is convenient to tabulate the factor $\frac{6.5\,m}{fk^2}$ in the form of a single engine constant, as in the following table:

Diameter of Wheel in Feet	Max. R. P. M. for Rim Speed of 4800 ft. / min.	Max. R. P. M. for Rim Speed of 5700 ft. / min.	Values of Engine Constant $\frac{65\,m}{fk^2}$ in Formula $W=\left(\frac{100}{N}\right)^2\left(\frac{65\,m}{fk^2}\right)$ kw.		
			Single Cylinder	Tandem Compound	Cross Compound
10	152	181	270	200	135
11	139	165	225	170	110
12	127	151	185	140	90
13	117	140	160	120	80
14	109	130	135	100	70
15	101	121	120	90	60
16	95	113	105	80	50
18	85	101	85	65	40
20	76	91	65	50	30

For A. C. in parallel operation, multiply the given tabular value of the engine constant by 2.

In shearing and punching machines it is customary to design the flywheel so that it will store energy equal in amount to the work done in two working strokes of the shear or punch, amounting to 17 to 20 inch tons per square inch of metal sheared or punched.

PROBLEMS

400. Calculate the radius of gyration of the flywheel shown in Fig. 255, there being six straight spokes, elliptical in cross section, $3\frac{1}{2}$ in. × 6 in.

401. The flywheel shown in Fig. 255 is on a 15 in. × 18 in. single cylinder engine running at 225 r. p. m. and operating a 175 kw. 3 phase D. C. generator. Compare the actual weight of the flywheel with that calculated from the formula given in Art. 136.

402. A cross compound engine is direct connected to a 500 kw. 60 cycle A. C. generator, operating in parallel, and running at 90 r. p. m. Find the required weight and diameter of balance wheel.

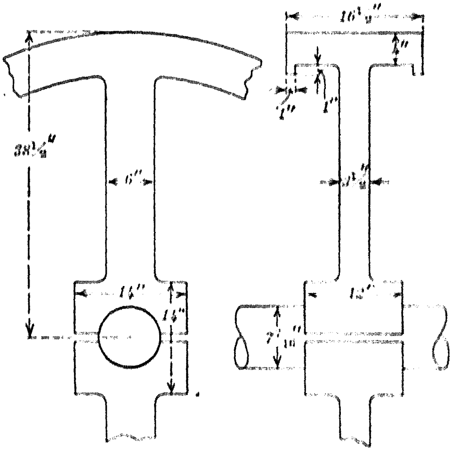


Fig. 255

SOLUTION.

By formula:

$k = 0.4 \times 20 = 8$ for $D = 20$. [illegible]

$f = 0.0025 = \frac{1}{4}$ per cent [illegible]

$W = \left(\frac{100}{20}\right)^2 \frac{[illegible]}{0.0025 \times [illegible]}$ = [illegible] lb.

By table:

[illegible] = 30; for A. C. in parallel = 60,

$W = \left(\frac{100}{20}\right)^2 \times 60 \times 300$ = 11,150 lb.

403. A single-cylinder engine is direct connected to a 75 kw. D.C. generator running at 120 r.p.m. Find the required diameter and weight of flywheel.

404. In an 8 h.p. Otto gas engine the speed is 264 r.p.m. Each fourth stroke is effective, and the allowable fluctuation in speed is 1 per cent on either side of the mean, i.e. $f = 0.02$. Find the required weight and diameter of balance wheel.

405. Find the required weight and diameter for the flywheel of a punching machine intended to punch a $1\frac{1}{2}$ in. hole in a $1\frac{1}{4}$ in. plate. Rim speed 30 ft./sec.

138. Rim Stresses. The stress in the rim of a flywheel is due to the centrifugal force caused by its own weight, and is a tensile stress tangential to the rim, or hoop stress, as it is called. To determine the amount of this hoop stress consider a circular ring under an internal unit pressure w normal to the surface. Taking a diametral section of the ring, as shown in Fig. 256, and considering an element of length l measured parallel to the axis of the ring, the total internal pressure perpendicular to the plane of the section

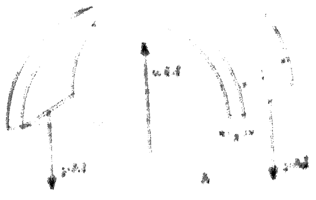


FIG. 256.

is wld, and consequently the tension in the ring itself is $\frac{1}{2}wld$ or wlr. If the rim is of thickness h and the unit tensile or hoop stress acting on it is denoted by p, then the total tensile stress on the cross section considered is phl, and equating this to the above, we have $phl = lwr$,

whence
$$p = \frac{wr}{h}.$$

In the case of a flywheel rim, the internal pressure w is the centrifugal force developed per unit length of rim. Hence if W denotes the weight of a unit length of the rim, we have

$$\text{Centrifugal force per unit length of rim} = w = \frac{Wv^2}{gr},$$

and hence the hoop stress p is, in the present case,

$$\text{Hoop stress in rim} = p = \frac{Wr^2\omega^2}{gh} = \frac{Wv^2}{gh},$$

where v denotes the rim speed in ft./sec.

To obtain the numerical value of the rim stress it is convenient to consider a section of the rim one square inch in area and one foot long, the former unit being chosen because the unit stress is expressed in lb./in.^{2}, and the latter because the rim speed v is expressed in ft./sec. For cast iron the weight of such an element is 3.1 lb. Hence

$$\text{Rim stress in lb./in.}^2 = \frac{3.1\,v^2}{32.2} = \frac{v^2}{10} \text{ approximately.}$$

For a rim speed of 5700 ft./min. this gives a rim stress of 900 lb./in.^{2}, which corresponds to a factor of safety of about 21.

Since good practice requires a factor of safety of 20 for cast iron, it is evident that there is a rational basis for the old practical rule that the rim speed should not exceed a mile a minute.

139. Tension in Spokes.—Since the spokes of a flywheel transmit the energy of rotation from the hub to the rim, they act like cantilever beams, fixed at the hub end and loaded at the outer end. In addition to the flexural stress to which they are thus subjected, it is also usual to design them to resist the tension which each spoke would have to carry if the rim was cracked so that all the centrifugal force developed would come on the spokes.

To find the amount of this tension, let

W = weight of the rim in pounds,
v = rim speed in ft./sec.,
r = radius of wheel in feet,
A = cross-section area of one spoke in square inches,
n = number of spokes.

Then the centrifugal force C arising from the rim alone is

$$C = \frac{Wr\omega^2}{g} = \frac{Wv^2}{gr},$$

and the amount carried by each spoke is $\dfrac{Wv^2}{grn}$. Consequently, the unit tensile stress p in the spokes is

$$p = \frac{Wv^2}{grnA}.$$

PROBLEM

406. Calculate the centrifugal tension in the spokes of the flywheel shown in Fig. 255 at the given speed of 225 r. p. m.

140. Bending Stress in Spokes. — Since the spokes transmit the energy of rotation from the hub to the rim, or *vice versa*, any fluctuation in speed will produce a bending stress in the spokes. Now the surplus energy E stored in the wheel per stroke is

$$E = I\omega^2 f,$$

or, considering the rim alone, if W denotes its weight and r its mean radius, then $I = \frac{Wr^2}{g}$, and consequently

$$E = \frac{Wr^2\omega^2 f}{g}.$$

Now considering each spoke as a cantilever beam fixed at the hub end and carrying a single concentrated load P at the outer end, if D denotes the deflection at the outer end, the work done in bending all n spokes this amount is $\frac{1}{2} PDn$. Equating this to the excess energy stored per stroke,

$$\frac{1}{2} PDn = \frac{Wr^2\omega^2 f}{g}.$$

To find the bending stress in each spoke make use of the ordinary beam formula, namely $p = \frac{Me}{I_0}$, where

p = unit bending stress in lb./in.^{2},
M = external bending moment in in.lb.
e = distance from neutral axis to extreme fiber,
I_0 = static moment of inertia of cross section of beam with respect to its neutral axis.

In the present case if l denotes the length of a spoke, we have $M = Pl$, and hence

$$P = \frac{pI_0}{le}.$$

It is well known, however, that the deflection of a cantilever of length l bearing a single concentrated load P at the end is

$$D = \frac{Pl^3}{3\,E_0 I_0},$$

where E_0 = Young's modulus of elasticity in lb./in.^{2}, and substituting the value of P in this expression, we have

$$D = \frac{Ml^2}{3\,E_0 I_0}.$$

Substituting these values in the expression for the work as given above, it becomes

$$\frac{Wr^2\omega^2 f}{gn} = \frac{1}{2} \times \frac{pI_0}{le} \times \frac{Ml^2}{3\,E_0 I_0},$$

or, since $M = \frac{pI_0}{e}$, this may be written

$$\frac{Wr^2\omega^2 f}{gn} = \frac{1}{2} \times \frac{pI_0}{le} \times \frac{pI_0}{e} \times \frac{l^2}{3\,E_0 I_0},$$

whence, solving for p, we have finally

$$p = v\sqrt{\frac{6\,WfE_0}{I_0 ngl}},$$

which is the required expression for the bending stress in the spokes in terms of the rim speed v, and its fluctuation f.

If there is a sudden stoppage of the engine, the energy destroyed is $\frac{1}{2}\,I\omega^2$, or, for the rim alone, $\frac{1}{2}\,\frac{W}{g}\,r^2\omega^2$. Consequently the energy absorbed by each spoke is $\frac{Wr^2\omega^2}{2\,gn}$, which is $2f$ times as great as that given above for ordinary fluctuations in speed. Hence the stress in this case is $\sqrt{\frac{1}{2f}}$ times as great as under ordinary working conditions. Since $\sqrt{\frac{1}{2f}}$ varies from $\sqrt{\frac{20}{2}}$ to $\sqrt{\frac{400}{2}}$, the stress due to a sudden stop is from 3 to 14 times as great as the ordinary working stress.

PROBLEM

407. A flywheel [illegible] ft. in diameter weighs [illegible] lb., of which [illegible] lb. is in the rim. Rim speed [illegible] ft./min. There are eight spokes of elliptical cross section [illegible] sq. in. [illegible]. Find the total stress in the spokes due to fluctuation in speed and to centrifugal force, and the factor of safety. Find also the stresses developed in the spokes by an instantaneous stop of the shaft.

141. Design of Flywheels. In designing flywheels there are certain general rules which must be followed in order to secure proper proportions.

The shaft on which the wheel is carried is the most convenient starting point. This determines the diameter of the bore. The outside diameter of the hub should then be not less than twice the diameter of the bore, and the length of the hub about $1\frac{1}{2}$ times the bore.

The next step is to calculate the total weight of the wheel by the formulas given above.* This is of course determined by the horsepower of the engine and the class of service required of it. The rim may then be designed by assuming that about 0.7 of the total weight of the wheel is carried in the rim, this being about the usual proportion in actual practice. The rim should in no case be less than $\frac{3}{4}$ in. thick, which limits the diameter.

The spokes may next be designed to carry the centrifugal tension and bending stress due to fluctuation in speed, by means of the formulas given above. A simpler way of designing the

* For single-cylinder engines the following empirical rules are used by engine builders:

$$W = k\frac{d^2 s}{D^2 N^2}$$

W = weight of wheel in pounds,
d = diameter of cylinder in inches,
D = diameter of flywheel in feet,
N = revolutions per minute,
s = length of stroke in inches,
k = empirical constant.

For slide valve engines, ordinary duty,	k =	350,000
For slide valve engines, electric lighting,	k =	700,000
For automatic high speed engines,	k =	1,000,000
For Corliss engines, ordinary duty,	k =	700,000
For Corliss engines, electric lighting,	k =	1,000,000

spokes is to consider each as a cantilever beam and determine its size so that it will have the same factor of safety against bending stress as the shaft has for torsional stress. That is to say, design the spokes so that they will be as strong considered as a cantilever beam as the shaft is to resist torque.

Having completed the design of the various parts, the actual total weight should be checked up with the required weight as first computed.

142. Governors. — As explained in the preceding articles, the purpose of a flywheel is to provide for changes in speed by storing kinetic energy; that is to say, its object is to decrease the effect of fluctuations in speed on the smooth running of an engine by calling into play inertia forces, chiefly through the mass of its rim. The object of a governor, on the other hand, is to eliminate changes in speed so far as possible by regulating the admission of steam to the cylinders. This is accomplished by using the centrifugal or inertia force of rotating balls or arms to open or close the valves. The arrangement of several of the more common forms of governor are shown in the following articles, and their effects analyzed and compared.

143. Watt Governor. — The simplest form of governor is that known as the Watt governor, which consists of a pair of balls suspended from a fixed point and rotating about a vertical axis (Fig. 257). As the speed increases the centrifugal force on the balls also increases, causing them to fly farther out from the axis of rotation and thereby partially closing the admission valve. If the speed decreases, due to increase in load or other causes, the balls fall in toward the axis again, thereby opening the valve.

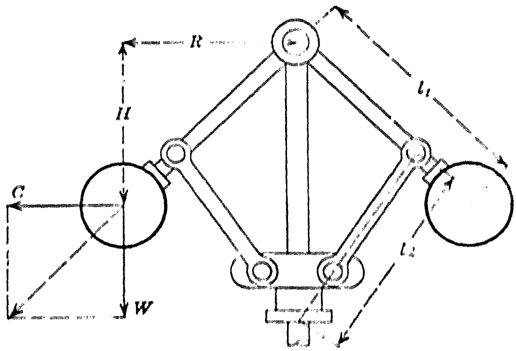


Fig. 257

Let W = weight of one ball in pounds,

C = centrifugal force on ball in pounds,

H = height of governor in feet,

h = height of governor in inches,

R = distance of ball from axis in feet,

r = distance of ball from axis in inches,

N = number of revolutions per second,

n = number of revolutions per minute.

Then, from Fig. 257,

$$\frac{H}{R} = \frac{W}{C} = \frac{W}{\frac{WR\omega^2}{g}} = \frac{g}{R\omega^2},$$

whence

$$H = \frac{g}{\omega^2}.$$

Since $\omega = 2\pi N$, this expression for H may be written

$$H = \frac{g}{4\pi^2 N^2} = \frac{0.816}{N^2}.$$

It is usually more convenient to express the height of the governor in inches and the speed in revolutions per minute. In these units the expression for the height becomes

$$h = \frac{35230}{n^2}.$$

From this expression it is evident that the height of a Watt governor is independent of the weight of the balls or their length of arm. The balls will thus remain in their lowest position until the speed n has increased so that the actual measured height of the governor is equal to the value of h just obtained. The balls will then begin to float, and upon further increasing the speed will fly out from the axis, the height h in each case corresponding to the speed n in accordance with the above relation.

The heights and changes in height corresponding to various speeds and increase in speed are shown in the following table:

R. P. M	Height h in Inches	Change in Height in Inches corresponding to Change in Speed of 10 R. P. M.
50	14.09	—
60	9.79	4.30
70	7.19	2.60
80	5.51	1.68
90	4.35	1.16
100	3.52	0.83
110	2.91	0.61
120	2.45	0.46
130	2.09	0.36
140	1.80	0.29
150	1.57	0.23

In Fig. 258 this relation is plotted in the form of a curve, showing the relation of height to speed. Since the left end of the curve is much steeper than the right, it is evident that there

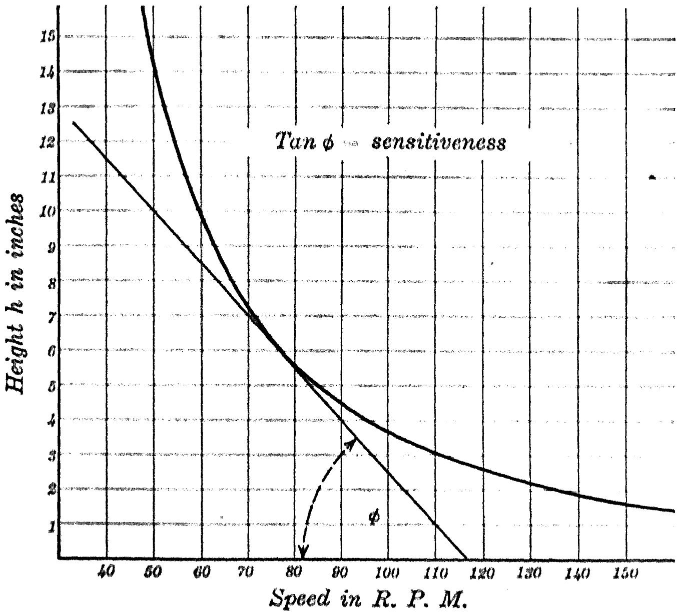


Fig. 258

will be more vertical motion of the governor, and consequently a greater travel of the valve, for a given change in speed if the

height is large, or, what amounts to the same thing, if the speed is low.

The slope of this curve, or ratio of a given change in height to the corresponding change in speed, is called the **sensitiveness** of the governor. To make a governor sensitive, therefore, it is only necessary to keep the height large by gearing down the governor so that it will run slowly. There is a limit to how far this can be carried, however, as decreasing the speed of the governor also decreases its power to overcome resistance, and consequently if the speed is too low it may be unable to lift the valve.

144. Loaded Governor. The power of a governor to overcome resistance, such as the friction on the sleeve, is increased by adding a central weight, which increases the downward pull on the balls without increasing the centrifugal force.

Fig. 250

A simple form of loaded governor is shown in Fig. 250 (a), in which the central weight is applied directly to the balls, which in this case are in the form of rollers upon which the disk rests.

Let P denote the central weight so applied. Then, using the same notation as before,

$$\frac{H}{R} = \frac{W + \frac{P}{2}}{C},$$

whence

$$H = \frac{0.816}{N^2} \cdot \frac{W + \frac{P}{2}}{W},$$

and

If the ratio of P to W is denoted by m, i.e. $P = mW$, the expression for h simplifies into

$$h = \frac{35230}{n^2}\left(1 + \frac{m}{2}\right).$$

The value of m in this formula usually lies between 10 and 50.

Comparing this expression for h with that for the unloaded governor, it is evident that if the governors are run at the same speed, the height of the loaded governor will be $1 + \frac{m}{2}$ times the height of the unloaded governor, and hence, neglecting friction, the loaded governor will be $1 + \frac{m}{2}$ times as sensitive as the unloaded.

If, however, the heights of the two governors are to be kept the same, the speed of the loaded governor must be $\sqrt{1 + \frac{m}{2}}$ times the speed of the unloaded. That is to say, if the engine is to run at the same speed with either governor, the loaded governor must be geared up this amount. Neglecting friction, the sensitiveness of the two governors, or travel of the valve for a given change in speed, will then be exactly the same, but as the loaded governor runs faster than the unloaded, it exerts a greater lifting effort.

When the central load is carried by means of links, as in the type known as the Porter governor, shown in Fig. 259 (b), the load rises faster than the balls and consequently does more work for a given change in height than if applied directly to the balls, as in Fig. 259 (a). Let s denote the ratio of the travel of the sleeve to the vertical motion of the balls. For example, if the arms and links are of equal length, the sleeve rises twice as fast as the balls, and $s = 2$. Then the expression for h becomes

$$h = \frac{35230}{n^2}\left[\frac{W + \frac{sP}{2}}{W}\right] = \frac{35230}{n^2}\left(1 + \frac{sm}{2}\right).$$

The simplest method for finding the ratio s is by means of a diagram. Thus in Fig. 260 (a), let CA denote the arm of the governor and AB the link connecting it to the sleeve. Produce CA to meet a horizontal line through B in the point O. Then O

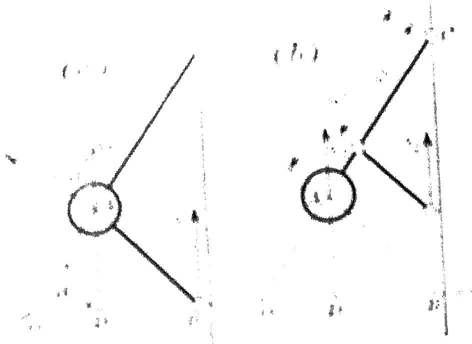


FIG. 260

is the instantaneous center for the link AB, and hence the ratio of the speed of the ends A and B of the link is

$$\frac{v_1}{v} = \frac{OB}{OA}.$$

Now let v_2 denote the vertical component of v, *i.e.* the speed with which the balls rise. Then

$$v_2 = v \cos \alpha,$$

and since $OA \cos \alpha = OD$, the above ratio becomes

$$s = \frac{v_1}{v_2} = \frac{OB}{OD}.$$

If the link is connected as shown in Fig. 260 (b), draw AB parallel to the link and complete the rest of the construction as above. Then

$$s = \frac{v_1}{v_2} = \frac{OB}{OD} = \frac{l_1}{l_2}.$$

145. Isochronous Governor. It is possible to construct a governor so sensitive that it will go through its entire range of travel for a very slight variation in speed on either side of a certain fixed or critical speed. That is to say, a slight increase in speed over that at which the balls begin to float will cause them to fly out to their full extent, while a slight decrease below the floating speed will cause them to fall back toward the axis as far as their construction will permit. Such an extremely sensitive governor is called **isochronous**.

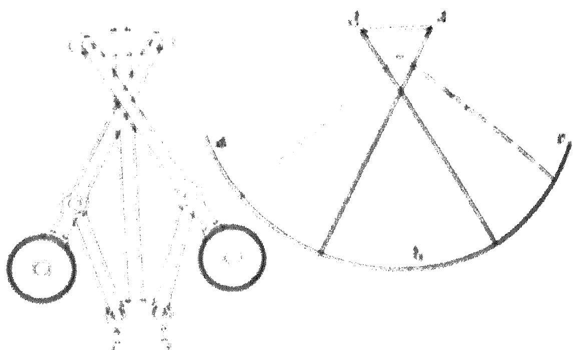


FIG. 261

The crossed arm type of isochronous governor is shown in Fig. 261. To explain its construction assume that a parabolic arc *abc* is laid off, and normals drawn to this arc at the highest and lowest positions of the balls. Let *d*, *d* be the points in which the two normals for each ball intersect. Then if the arms of the governor are pivoted at the points *d*, *d*, the balls will move in an approximately parabolic arc. Now it is a property of the parabola that its subnormal is constant (Fig. 262). This means in the present case that the height of the governor is constant, and therefore that n is also constant, whether the governor is loaded or not. Consequently there is only one speed at which the balls will float, and, neglecting friction, a slight variation in speed will cause motion to the full extent of the entire travel of the valve in either direction.

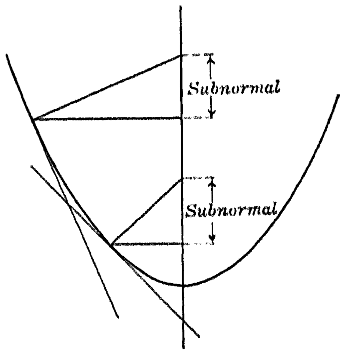


FIG. 262

146. Crank Shaft Governor. — The governing of an engine is frequently effected by changing the point of cut-off in the cylinder. This is done by changing the throw of the eccentric which operates the valve, and to accomplish this the crank shaft governor is now generally used. This type of governor has the further advantage of being practically isochronous.

The crank shaft type of governor is shown in Fig. 263. This consists in outline of two bars AB and CD, pivoted at P_1, P_2, with centers of gravity at G_1, G_2, and connected by a link AC (Fig. 264). The lower bar CD has its center of gravity practically at the center of the shaft, so that it generates no centrifugal force. Its object is to balance the static moment of the bar AB in all positions. Otherwise the weight of AB would have an upward moment about P_1 during one half of a revolution and a downward moment the other half, thus affecting the governing.

Now let W denote the weight of AB, and r the distance from its center of gravity to the center of the shaft. Then the cen-

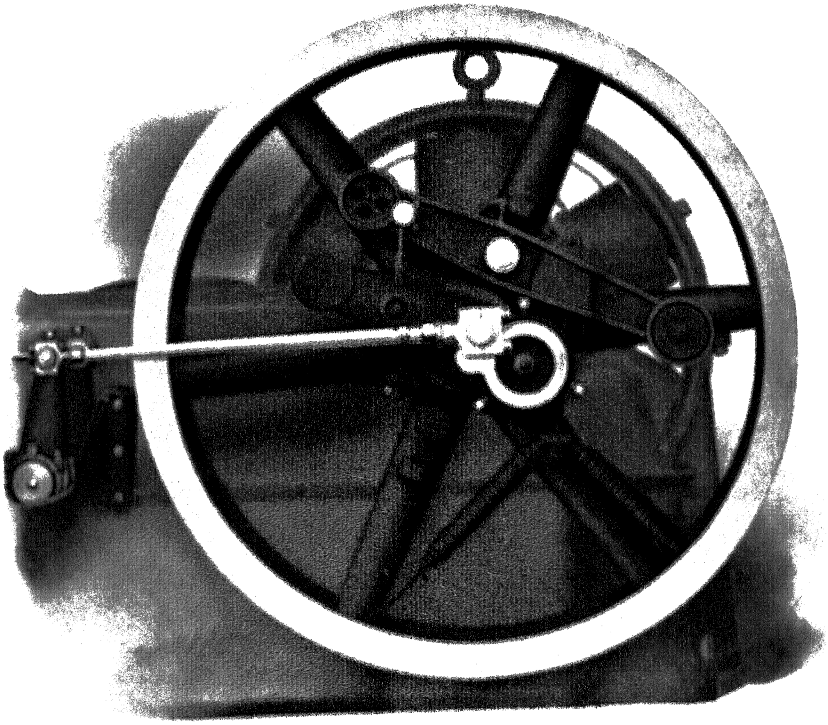


FIG. 263

trifugal force developed at the speed ω is

$$C = \frac{W}{g} r\omega^2,$$

FIG. 264

or, since $\omega = \frac{2\pi n}{60}$, this becomes

$$C = \frac{4\pi^2 n^2 r W}{3600\, g},$$

which reduces to

$$C = 0.000341\, Wrn^2.$$

When the moment of this force about the pivot P_1 just equals the moment of the tension in the spring about P_1, the governor will begin to act.

This type of governor is so sensitive that its makers claim that it will govern within ½ of 1 per cent.

PROBLEM

408. In a crank shaft governor the weight of the centrifugal arm is $W = 60$ lb., the distance of its center of gravity from the center of the shaft is $r = 8$ in., the tension in the spring is 500 lb., the lever arm of the centrifugal force about the pivot P is 4.5 in., and the lever arm of the tension in the spring about P is 8 in. At what speed will the governor begin to act?

147. Governor Effort and Power. — By the effort of a governor is meant the force it is capable of exerting on the sleeve for a given percentage of change in speed.

For a simple Watt governor $h = \frac{g}{\omega^2}$, and for a loaded governor $h = \frac{g}{\omega^2}\frac{W+P}{W}$, whence

$$\omega^2 = \frac{g}{h}\frac{W+P}{W}.$$

Now suppose that the speed is increased to $q\omega$, and that the balls are prevented from rising immediately to the height corresponding to this speed by a frictional force F acting downward on the sleeve, which is equivalent to increasing the central load to $P + F$.

The centrifugal force on each ball corresponding to the speed ω is

$$C = \frac{W\omega^2 r}{g},$$

and at the speed $q\omega$, since the balls do not rise and hence the radius r remains the same, this becomes

$$C_1 = \frac{Wq^2\omega^2 r}{g},$$

and consequently, $$C_1 - C = \frac{Wr\omega^2}{g}(q^2 - 1).$$

Moreover, equating the moment of the centrifugal force about the point of suspension to the moment of the weight about this point, we also have

$$Ch = (W + P)r,$$

and also since the height and radius both remain the same,

$$C_1 h = (W + P + F)r,$$

and consequently $$C_1 - C = \frac{rF}{h}.$$

Equating the two values of $C_1 - C$ so obtained, we have

$$\frac{Wr\omega^2}{g}(q^2 - 1) = \frac{rF}{h},$$

whence $$F = \frac{W\omega^2 h}{g}(q^2 - 1) = (W + P)(q^2 - 1).$$

Since the frictional resistance only acts while the sleeve is in motion, *i.e.* it becomes zero when the sleeve ceases to rise, the average value of the effort on the sleeve during the rise is $\frac{1}{2}F$. Hence the resistance R at the sleeve which this type of governor is capable of overcoming for an increase in speed from ω to $q\omega$ is

$$R = \frac{W + P}{2}(q^2 - 1).$$

For example, suppose that the speed changes one per cent, *i.e.* changes from ω to $1.01\ \omega$. Then $q = 1.01$ and $\frac{1}{2}(q^2 - 1) = \frac{1}{2}(1.02 - 1) = 0.01$. Consequently the allowable resistance at the sleeve for this increase in speed is $0.01(W + P)$.

For a simple or unloaded governor $P = 0$, and hence if the arms and links are of equal length, the effective effort in this case is

$$R = \frac{W}{2}(q^2 - 1).$$

If the sleeve is suspended as shown in Fig. 257 and the arms and links are equally inclined to the axis,

$$R = \frac{W}{2}(q^2 - 1)\frac{l_1}{l_2}.$$

The amount of work which the governor is capable of doing on the resistance at the sleeve during the given change in speed is called the power of the governor.

For the loaded governor the height corresponding to the speed ω is

$$h = \frac{g}{\omega^2} \cdot \frac{W + P}{W},$$

and to the speed $q\omega$ is $h_1 = \frac{g}{q^2\omega^2} \times \frac{W+P}{W}$.

Consequently the change in height Δh corresponding to an increase in speed from ω to $q\omega$ is

$$\Delta h = h - h_1 = \frac{g}{\omega^2}\,\frac{W+P}{W}\left(1 - \frac{1}{q^2}\right) = h\left(\frac{q^2-1}{q^2}\right).$$

The work V done on the resistance R during the given change in speed is, then,

$$V = R \times 2\,\Delta h = \frac{W+P}{2}(q^2-1)\,2h\left(\frac{q^2-1}{q^2}\right) = (W+P)\,h\left(\frac{q^2-1}{q}\right)^2.$$

For the simple Watt governor (Fig. 257)

$$R = \tfrac{1}{2}W(q^2-1)\frac{l_1}{l_2} \text{ and } \Delta h = \left(\frac{q^2-1}{q^2}\right)h.$$

Consequently in this case

$$V = R \times 2\,\Delta h\frac{l_2}{l_1} = Wh\left(\frac{q^2-1}{q}\right)^2.$$

148. Sensitiveness of Governors.—The sensitiveness of any given type of governor may be shown graphically by constructing the curves representing the centripetal and centrifugal moments. For this purpose let W denote the weight of one ball of a simple Watt governor in pounds, and R its distance from the axis of rotation in feet. Then the centrifugal force acting on the ball at a speed of N revolutions per minute is $C = 0.00034\ WRN^2$. Consequently, if H denotes the height of the governor in feet, the centrifugal moment T about the point of suspension O, in foot pounds, is

$$T = CH = 0.00034\ WRN^2H.$$

Hence, if T is calculated for various positions of the governor, that is, for pairs of values of R and H, and is then laid off as an ordinate directly beneath each position of the ball, a centrifugal moment curve is obtained for each speed, as shown in Fig. 265.

Since the centripetal moment is simply $T = WR$, the centripetal moment curve is a straight line. The intersections of this line with the centrifugal moment curves show at what speeds the

valve will open and close. Thus in Fig. 265 let the positions shown represent the extreme positions of the governor. Then, noting the intersections, it is evident that the governor will open at about 88 r. p. m., and close at 96 r. p. m. The mean speed at which the governor will run is, therefore, 92 r. p. m., and the variation in speed is about 4.4 per cent on either side of the mean.

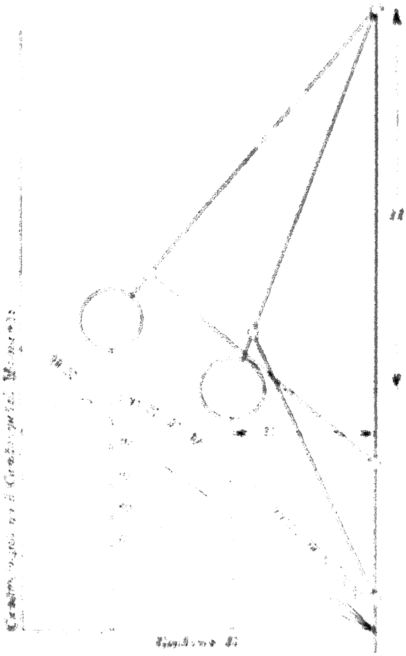


Fig. 265

This result, however, is modified by frictional resistance, which has the same effect as adding to the weight when the balls are rising, and subtracting from it when they are falling. Consequently the effect of friction is to add or subtract a constant amount from the centripetal moment, which is represented graphically by drawing parallels on either side of the line representing the centripetal moment, the vertical distance between the centripetal moment line and either of these parallels being equal to the frictional moment. In Fig. 266 the shaded strips represent the effect of the frictional moment. In the particular case here shown the governor now begins to rise at 94 r. p. m., and in falling the speed drops to 80 r. p. m. Hence the variation in speed is now 13 per cent on either side of the mean. From this is evident the great importance of eliminating friction as much as possible in designing a governor.

Fig. 266

For a crossed arm governor the height H is constant. Hence, for this type of governor the centrifugal moment varies directly as R, and consequently the centrifugal moment curves are straight lines. Since one of these lines must coincide with the line representing the centripetal moment, it is evident that at the corresponding speed there will be nothing to prevent the balls from flying entirely out or falling entirely in, which is, in fact, the reason why this type of governor is isochronous.

For a crank shaft governor the centrifugal moment curves are very nearly straight lines. Hence the centripetal moment line very nearly coincides with some one of these curves, and consequently this type of governor is very nearly isochronous.

149. Piston Acceleration.—It has previously been shown that if the speed of the crank pin of an engine is constant and the connecting rod is assumed to be infinitely long, the speed curve for the piston, or cross head, is a semicircle, and the acceleration curve is a straight line (Chap. I, Art. 28, and Prob. 71, Art. 32). The actual shape of the piston speed curve for a connecting rod of given length has also been found, and its equation deduced (Art. 32, including Prob. 70).

In the dynamics of the steam engine it is a sufficient approximation for most purposes to assume that the acceleration of the reciprocating parts is represented by a straight line, which is equivalent to neglecting the obliquity of the connecting rod, or assuming it to be of infinite length. The actual shape of the acceleration curve, however, may be obtained as follows:

In drawing the piston speed curve, the distance from the middle of the stroke is laid off as abscissa, and the corresponding speed as ordinate; that is to say, the general form of the piston speed curve is

$$v = f(x).$$

Now in the case of any curve whose equation is $y = f(x)$, the subnormal at any point is given by the expression

$$\text{Subnormal} = y\frac{dy}{dx},$$

as shown in Fig. 267. In the case of the piston speed curve the

subnormal at any point is, therefore, $v\frac{dv}{ds}$, or separating this into partial derivatives with respect to the time,

$$\text{Subnormal} = v\frac{dv}{ds} = v\frac{\frac{dv}{dt}}{\frac{ds}{dt}} = \frac{v\frac{dv}{dt}}{v},$$

$$= \frac{dv}{dt} = \text{acceleration}.$$

Hence we have the theorem that *the subnormal to the piston speed curve represents the piston acceleration.*

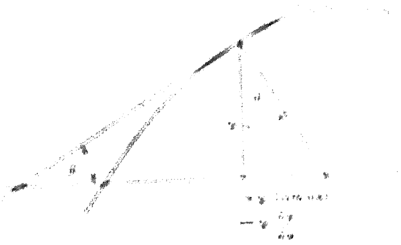


FIG. 267

Although this apparently affords a simple method for the graphical construction of the piston acceleration curve, it is not a practical solution because the subnormal takes the indeterminate form $\frac{0}{0}$ at either end of the stroke; and also because the accuracy of the construction depends on that of the piston speed curve, and the result is therefore not likely to be more accurate than the simple assumption of a straight line.

The most accurate graphical construction of the piston acceleration curve is that known as Klein's construction, which will now be explained. The proof of this construction is not given, as it is long and somewhat complicated.

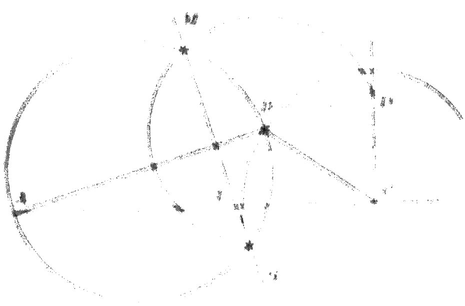


FIG. 268

Let BC denote the crank and AB the connecting rod (Fig. 268). On AB as diameter construct a circle. Prolong AB until it intersects the vertical through C (*i.e.* perpendicular to the line of stroke) in the point D. With B as center

and BD as radius describe another circle. Draw the common chord MN of these two circles. Then the intercept LC on the line of centers represents the piston acceleration for the position shown.

It will be noted that in this construction the intersection at L is never sharply acute, and consequently the intercept LC, or acceleration, can always be accurately determined. Moreover, it does not depend on any previous construction, which also insures accuracy.

At the ends of the stroke the construction becomes as shown in Fig. 269. To determine the numerical values of the acceleration at the ends of the stroke, let v denote the linear speed of the

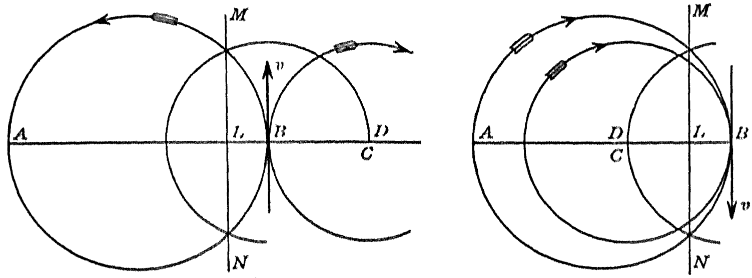


Fig. 269

crank pin, r the length of the crank, and l the length of the connecting rod. Then at the out end of the stroke the angular velocity of the connecting rod is $\frac{v}{l}$ and the angular velocity of the crank is $\frac{v}{r}$. Since the normal acceleration in any case is given by the expression $a = r\omega^2$,

the crank pin B has therefore an acceleration in the direction AC of amount $\frac{v^2}{l}$, and in the direction CA of amount $\frac{v^2}{r}$. Its actual acceleration at the out end of the stroke is therefore $\frac{v^2}{l} - \frac{v^2}{r}$, or, if the ratio of the length of the connecting rod to the length of the crank is denoted by q, that is, $l = qr$, this becomes

$$\frac{v^2}{r}\left(1 - \frac{1}{q}\right).$$

Similarly, at the in end of the stroke the crank pin B has an acceleration $\frac{v^2}{l}$ in the direction AB and an acceleration $\frac{v^2}{r}$ in the same direction. Hence the total acceleration of B is $\frac{v^2}{l} + \frac{v^2}{r}$, or since $l = qr$, this becomes

$$\frac{v^2}{r}\left(1 + \frac{1}{q}\right).$$

PROBLEM

409. By the method given in Art. [illegible] [illegible] Klein's construction, as just explained, construct on the same base the piston speed and acceleration curves for a crank [illegible] long, connecting rod [illegible] ft. long, and piston speed of 600 ft./min. (See Fig. [illegible].)

150. Inertia Thrust. The acceleration of the reciprocating parts of an engine produces an acceleration pressure, or inertia thrust, which seriously affects the running of the engine unless properly compensated. Since force is proportional to acceleration, the curve of inertia thrust is similar to the curve of piston acceleration. That is to say, by choosing the proper scale, the same curve may be used to represent both the piston acceleration and the inertia thrust on the reciprocating parts.

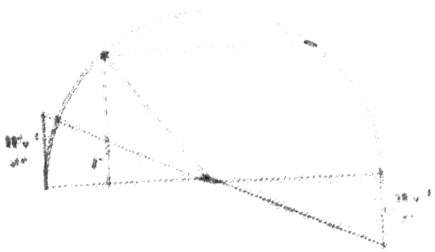


Fig. 279

For an infinitely long connecting rod, the acceleration at the ends of the stroke is $\frac{v^2}{r}$. Hence, if W denotes the weight of the reciprocating parts, the inertia thrust at the out end of the stroke is $\frac{Wv^2}{gr}$, and at the in end of the stroke is $\frac{Wv^2}{gr}$. At any intermediate position the inertia thrust P is

$$P = \frac{Wv^2}{gr} \cdot \frac{x}{r}$$

as indicated in Fig. 279.

For an ordinary connecting rod, i.e. one of finite length l, the results of the preceding article show that the inertia thrust at the

out end of the stroke is

$$P = \frac{Wv^2}{gr}\left(1 - \frac{1}{q}\right),$$

and at the in end of the stroke is

$$P = \frac{Wv^2}{gr}\left(1 + \frac{1}{q}\right),$$

as indicated in Fig. 271.

It should be noted that the point where the curve of inertia thrust crosses the base line is the point at which the piston speed curve attains a maximum, *i.e.* approximately the point at

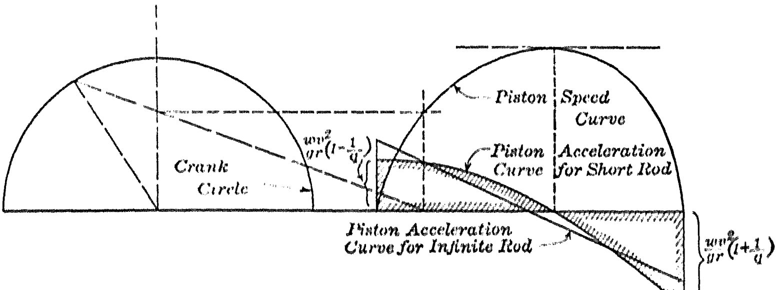


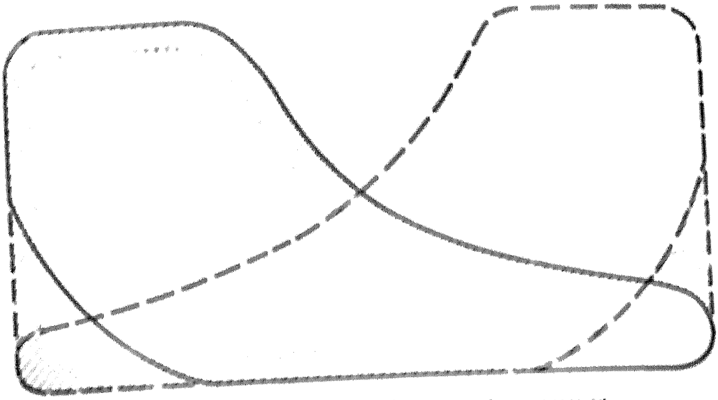
Fig. 271

which the crank is at right angles to the connecting rod. In drawing the inertia thrust curve, it is usually sufficient for practical purposes to locate this point and the two ends of the curve, given by $P = \frac{Wv^2}{gr}\left(1 \pm \frac{1}{q}\right)$, and then connect the middle point with the end ones by straight lines.

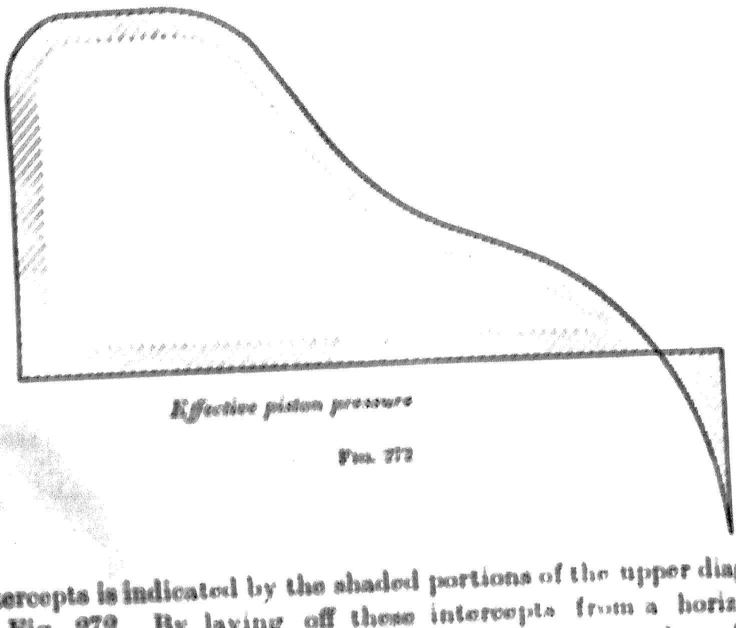
151. Piston Pressure. — In heat engines the pressure on the piston is, in general, variable throughout the stroke. In this case the pressure at any point of the stroke is obtained from an indicator card, drawn automatically by a self-recording pressure gauge (see Chap. II, Art. 43, Fig. 74). In the ordinary double-acting steam engine, two indicator cards are required for each cylinder, one for each side of the piston, as shown in the upper diagram of Fig. 272.

The effective steam pressure on the piston at any point of the stroke is measured by the vertical distance between the top line

of one card and the bottom line of the other, since this intercept represents the difference between the steam pressure on opposite sides of the piston at the point in question. The limit of these



Indicator cards showing effective piston pressure



Effective piston pressure

FIG. 272

intercepts is indicated by the shaded portions of the upper diagram in Fig. 272. By laying off these intercepts from a horizontal base line, a curve of effective piston pressure is obtained, as shown by the second diagram of Fig. 272.

From the beginning of the stroke to the point of maximum speed, part of the steam pressure on the piston is required to accelerate the piston and other reciprocating parts. The work so

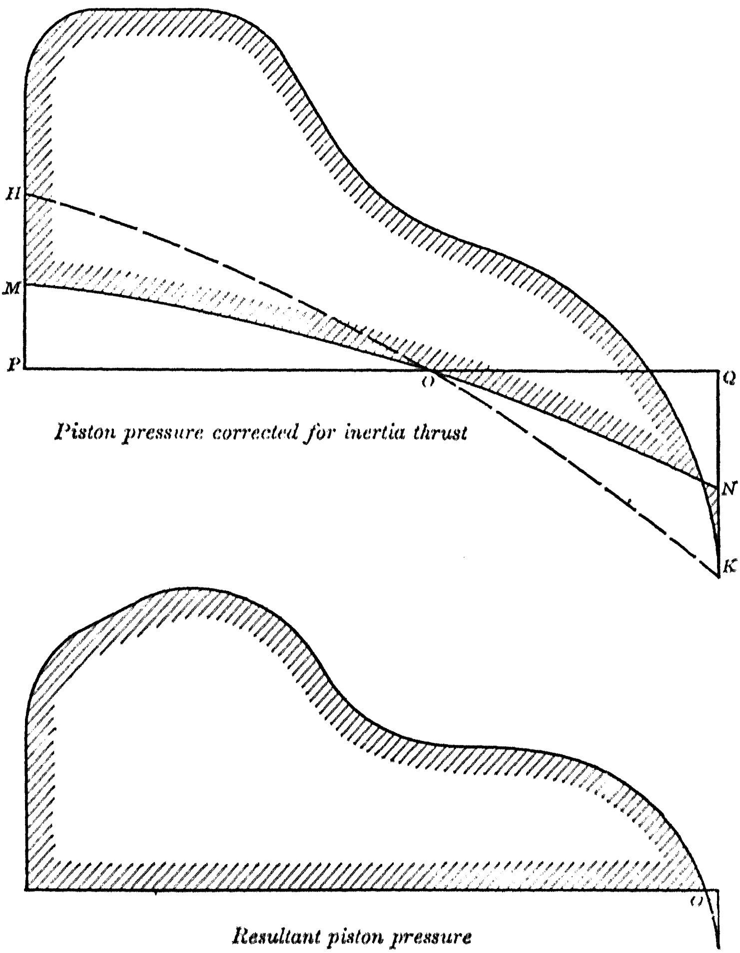


FIG. 272

done is stored up in these parts in the form of kinetic energy, and after the point of maximum velocity (or zero acceleration) is passed, this energy is given out again, thereby supplementing the work done by the steam during this part of the stroke.

To apply the correction for inertia thrust to the curve of effective piston pressure, plot both to the same scale on a horizontal base, as shown in the third diagram of Fig. 272. The curve of inertia thrust, HOK, is first obtained, say by Klein's construction, as explained in Art. 150. The ordinates to this curve are then reduced to the proper scale by altering them in the ratio of HP to MP, where MP represents the inertia thrust per square inch of piston at the end of the stroke; that is, if A denotes the area of the piston, and $\frac{W}{A} = w$ the weight of the reciprocating parts per square inch of piston, then

$$MP = \frac{wv^2}{gr}\left(1 + \frac{1}{n}\right).$$

The reduction may also be made from the other end of the diagram by changing the ordinates in the ratio of QK to QN, where similarly

$$QN = \frac{wv^2}{gr}\left(1 - \frac{1}{n}\right).$$

The vertical intercept between these two curves at any point of the stroke then represents the resultant pressure on the cross head actually transmitted to the connecting rod. Plotting these intercepts on a horizontal base, a curve of resultant piston pressure, or cross head pressure, is obtained, as shown by the lower diagram in Fig. 272.

If the curve of resultant piston pressure crosses the axis, as at O in the lower diagram of Fig. 272, the pressure is reversed at this point. The effect of this is to cause a knock in the engine at the point O, and also at the end of the stroke where the pressure suddenly changes from negative to positive. This may be obviated by diminishing the compression and initial pressure, or by increasing the speed. If the pressure curve crosses the axis near the opposite end, the initial pressure and compression are too small or the speed is too high.

Since the inertia thrust does not create energy, it is evident that it does not affect the mean piston pressure during the stroke, or the total amount of work done. Consequently the area of the

diagram of resultant piston pressure must be equal to the area of the original indicator card.

If the effective areas of the two sides of the piston are not equal, as, for instance, when one area is diminished by the cross section of the piston rod, it is necessary to alter the vertical scale of one indicator card before the above construction can be carried out. To make this correction, either the ordinates of the card for the smaller piston area must be reduced in the ratio of the smaller to the larger area, or the ordinates of the card for the larger area must be increased in the inverse ratio.

If the pressure scales are different, a similar correction must be applied. For example, in a triple-expansion engine, both the areas and the pressure scales differ for each cylinder. The indicator cards must therefore be altered so as to give equivalent pressures on pistons of equal area.

In vertical engines the weight of the piston and other reciprocating parts helps to accelerate the motion during the down stroke and retard it during the up stroke. Hence if w denotes the weight of the reciprocating parts per square inch of piston, the diagram of effective piston pressure must be corrected by lowering the base line an amount w for the down stroke, thereby increasing the ordinates by this amount, and raising it the same amount for the up stroke, thereby diminishing the ordinates by this amount.

PROBLEMS

410. To what point should compression be carried in the case of a horizontal engine running at 60 r. p. m., stroke 4 ft., weight of reciprocating parts 3.2 lb./in.^{2} of piston, length of connecting rod 9 ft.?

411. Trace the indicator diagrams shown in Fig. 273, and from them construct the curve of resultant piston pressure, assuming the weight of reciprocating parts to be 950 lb., crank radius 16 in., and length of connecting rod 6 ft. 8 in.

412. Construct the curve of resultant piston pressure from the indicator diagrams shown in Fig. 274, the weight of the reciprocating parts in this case being 800 lb., crank radius 15 in., and the length of connecting rod 6 ft.

413. Construct the curve of resultant piston pressure from the indicator cards shown in Fig. 275, assuming the length of the connecting rod to be 7 ft., crank radius $1\frac{1}{4}$ ft., and weight of the reciprocating parts 1000 lb.

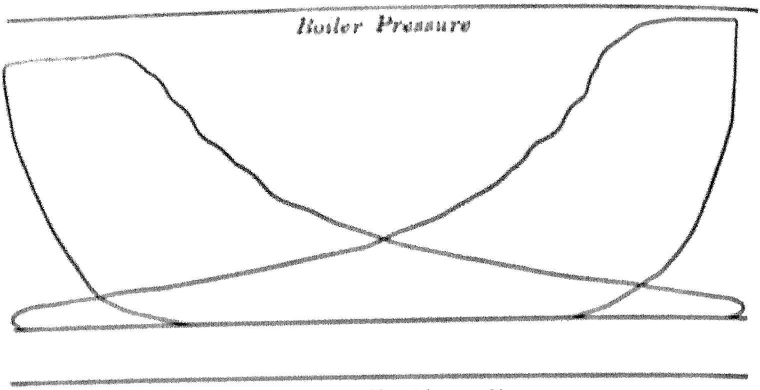


Diagram from Fitchburg Engine

Cylinder 20" diam., 40" stroke. Boiler pressure, 80 lb./in.². Speed, 90 R. P. M.

FIG. 273

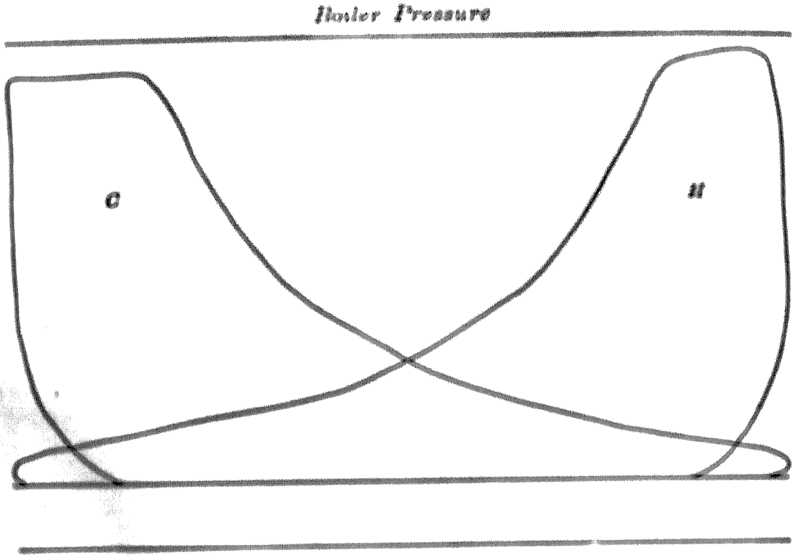


Diagram from Bates Corliss

Cylinder 18" diam., 36" stroke. Boiler pressure, 100 lb./in.². Speed, 90 R. P. M.

FIG. 274

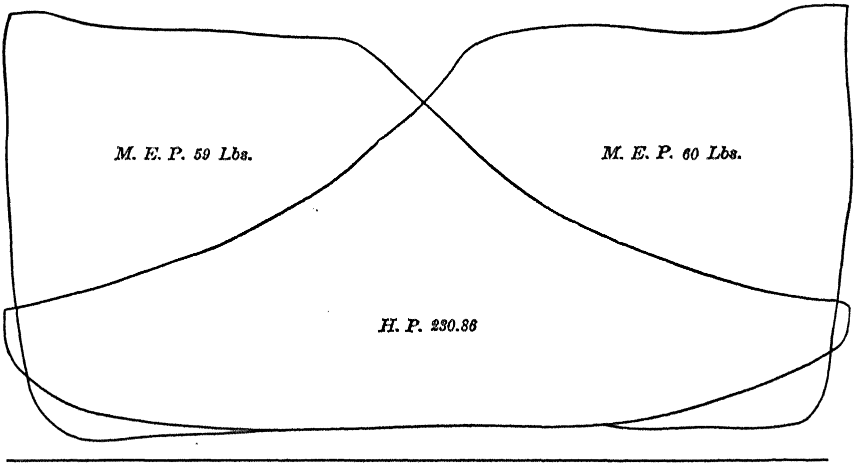


Diagram from 18 × 42-inch Corliss

Gauge pressure, 90 lb./in.². Speed, 73 R. P. M.

FIG. 275

152. Crank Pin Pressure. — Having found the force exerted at the cross head by the constructions explained in Art. 151, the next step is to find the effective force exerted on the crank pin. The product of this force by the radius of the crank circle will then be equal to the torque transmitted to the axle or shaft.

Let F denote the effective force exerted on the crank pin, that is, normal to the crank or tangential to the crank circle, and v_f the linear speed of the crank pin. Then the product Fv_f represents the power which the crank is transmitting at the given instant. To prove this, substitute for v_f its value as the derivative of the space with respect to the time, namely, $v_f = \frac{ds}{dt}$. Then $Fv_f = F\frac{ds}{dt}$. Therefore, since Fds represents an element of work, $\frac{Fds}{dt}$, or its equal Fv_f, represents the rate at which work is being performed, *i.e.* the power being transmitted at the instant considered.

Now let P denote the pressure on the cross head corresponding to the crank pin pressure F, and v_p its speed. Since, neglecting friction, the same amount of power is available at the crank pin

as at the cross head, it follows that $Pv_p = Fv_c$, or

$$\frac{F}{P} = \frac{v_p}{v_c};$$

that is to say, the ratio of the two forces is inversely as the ratio of their speeds.

In Art. 32, Chapter I, it was shown that if the speed of the crank pin is constant and is represented by the length of the crank CB (Fig. 276), then the intercept CD made by the connecting rod on a vertical through the center C of the crank circle represents the piston speed. From the relation just proved, however,

FIG. 276

$$\frac{F}{P} = \frac{v_p}{v_c} = \frac{CD}{CB}.$$

Hence if a distance CE is laid off along CB to represent the piston pressure drawn to any given scale, and a line EF is drawn through E parallel to the connecting rod, then CF will represent to the same scale the tangential thrust on the crank pin, since from similar triangles

$$\frac{F}{P} = \frac{v_p}{v_c} = \frac{CD}{CB} = \frac{CF}{CE}.$$

An analytical expression for the force exerted on the crank pin is also easily obtained from Fig. 276. Thus resolving the piston pressure P along the connecting rod AB, the thrust Q in the connecting rod is

$$Q = \frac{P}{\cos\alpha},$$

at the other end of the connecting rod B, resolving Q into tangential and normal components, the component F, tangential to the crank circle is

$$F = Q\sin(\alpha + \beta).$$

Substituting for Q its value in terms of P, this becomes

$$F = \frac{P\sin(\alpha + \beta)}{\cos\alpha} = P(\sin\beta + \cos\beta\tan\alpha).$$

To eliminate α from this expression, let l denote the length of the connecting rod, r the radius of the crank circle, and $l = qr$. Then from the figure

$$\frac{l}{r} = \frac{\sin \beta}{\sin \alpha} = q.$$

Hence
$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\sin \beta}{q\sqrt{1 - \frac{\sin^2 \beta}{q^2}}} = \frac{\sin \beta}{\sqrt{q^2 - \sin^2 \beta}},$$

and consequently
$$F = P\left(\sin \beta + \frac{\sin 2\beta}{2\sqrt{q^2 - \sin^2 \beta}}\right).$$

PROBLEMS

414. From the data and results of Prob. 411 draw a curve of resultant crank pin pressure.

415. Calculate from the data and results of Prob. 412, by means of the formula deduced above, the crank pin pressure when the crank has turned through an angle of 30° from the in end of the stroke.

153. Crank Effort Diagrams. — In all heat engines the force exerted on the crank pin is variable throughout the stroke. The construction given in the preceding article enables us to plot a diagram, showing graphically this fluctuation in crank effort.

To construct a crank effort diagram plot the curve of effective piston pressure, or cross head pressure, and also the crank circle, as shown in Fig. 277. The piston pressure curve in this figure is a duplicate of that shown in Fig. 272, and the present construction is therefore a continuation of that explained in Art. 151. Draw the crank in any convenient series of positions, as shown in the figure. By the construction given in the preceding article determine the crank effort F in these various positions, corresponding to the piston pressure P scaled from the piston pressure curve. This series of crank efforts may then be laid off so as to give either a rectangular or polar curve of crank effort. The former is shown in Fig. 277 (b), the base of the diagram being equal to $\frac{\pi d}{2}$ or the length of the path traversed by the crank pin

as at the cross head, it follows that $Pv_p = Fv_c$, or

$$\frac{F}{P} = \frac{v_p}{v_c};$$

that is to say, the ratio of the two forces is inversely as the ratio of their speeds.

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Fig. 276

$$\frac{F}{P} = \frac{v_p}{v_c} = \frac{CD}{CB}.$$

Hence if a distance CE is laid off along CB to represent the piston pressure drawn to any given scale, and a line EF is drawn through E parallel to the connecting rod, then CF will represent to the same scale the tangential thrust on the crank pin, since from similar triangles

$$\frac{F}{P} = \frac{v_p}{v_c} = \frac{CD}{CB} = \frac{CF}{CE}.$$

An analytical expression for the force exerted on the crank pin is also easily obtained from Fig. 276. Thus resolving the piston pressure P along the connecting rod AB, the thrust Q in the connecting rod is

$$Q = \frac{P}{\cos \alpha};$$

at the other end of the connecting rod B, resolving Q into tangential and normal components, the component F, tangential to the crank circle is

$$F = Q \sin(\alpha + \beta).$$

Substituting for Q its value in terms of P, this becomes

$$F = \frac{P \sin(\alpha + \beta)}{\cos \alpha} = P(\sin \beta + \cos \beta \tan \alpha).$$

To eliminate α from this expression, let l denote the length of the connecting rod, r the radius of the crank circle, and $l = qr$. Then from the figure

$$\frac{l}{r} = \frac{\sin\beta}{\sin\alpha} = q.$$

Hence $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sin\beta}{q\sqrt{1 - \frac{\sin^2\beta}{q^2}}} = \frac{\sin\beta}{\sqrt{q^2 - \sin^2\beta}},$$

and consequently

$$F = P\left(\sin\beta + \frac{\sin 2\beta}{2\sqrt{q^2 - \sin^2\beta}}\right).$$

PROBLEMS

414. From the data and results of Prob. 411 draw a curve of resultant crank pin pressure.

415. Calculate from the data and results of Prob. 412, by means of the formula deduced above, the crank pin pressure when the crank has turned through an angle of 30° from the in end of the stroke.

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during a single stroke, which makes the area of the diagram represent the work done during the stroke, and therefore equal to the area of the original indicator card.

To obtain the polar diagram, each crank effort is laid off along the corresponding radius to the crank circle, either entirely out-

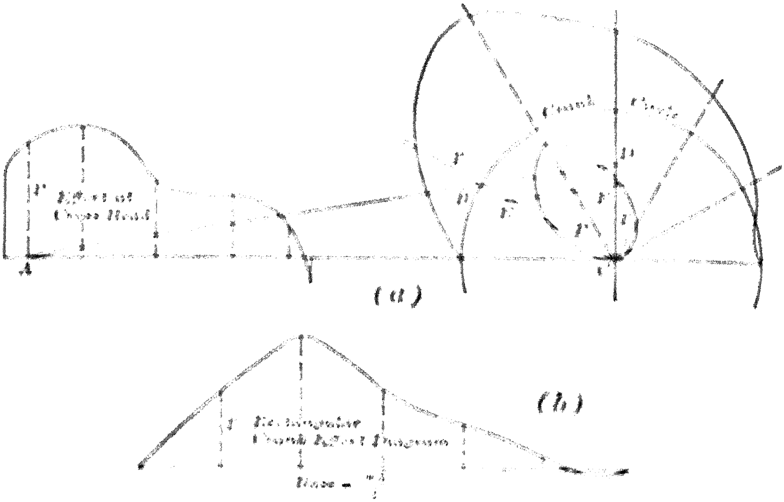


Fig. 277

side the latter and measured from the circumference, or radiating from the center C as origin, giving one or the other of the two polar diagrams shown in Fig. 277.

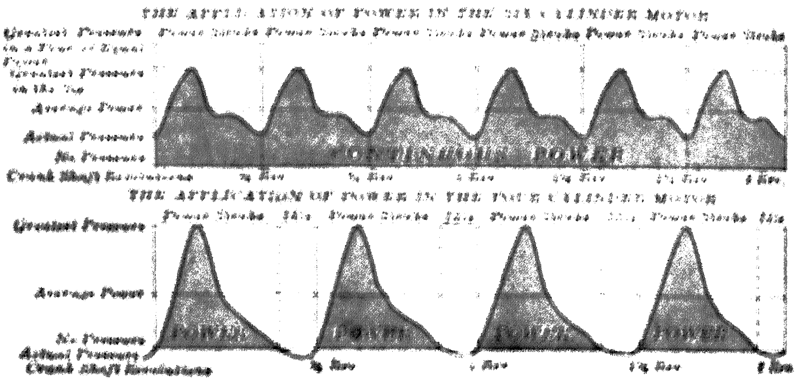


Fig. 278

As the number of cylinders is increased, the crank effort, and consequently the torque on the shaft, becomes much more uniform. Figure 278 shows rectangular diagrams of crank effort for a four-cylinder and for a six-cylinder automobile gas engine.

PROBLEMS

416. From the data and results of Prob. 413 construct a rectangular crank effort diagram.

417. In the preceding problem construct the polar crank effort diagrams.

154. Standing and Running Balance of Eccentric Weights. — When an eccentric weight revolves about a fixed axis such as a shaft in bearings, a centrifugal force is generated, acting outward from the axis of rotation toward the center of gravity of the weight. Let ω denote the angular velocity with which the eccentric weight W rotates, and r its eccentricity or distance out of center; *i.e.* the distance from the center of gravity of the weight to the axis of rotation. Then the centrifugal force C developed at the speed ω is

$$C = \frac{W\omega^2 r}{g}.$$

This centrifugal force may be balanced, or neutralized, by adding another eccentric weight in the same plane as the first weight and diametrically opposite to it. Let W' denote the weight of this counterbalance and r' the distance from its center of gravity to the axis of rotation, or eccentricity. Then the condition that the two shall balance when the shaft is at rest (standing balance) is that their mutual center of gravity shall lie in the axis of rotation, namely,

$$Wr = W'r'.$$

The condition that the centrifugal forces shall balance when the shaft is in motion is

$$\frac{W\omega^2 r}{g} = \frac{W'\omega^2 r'}{g},$$

or, cancelling out the common factor $\frac{\omega^2}{g}$, simply

$$Wr = W'r'.$$

Consequently if two or more eccentric weights rotate in the same plane, the conditions for standing balance and running balance are identical, namely,

$$\sum Wr = 0.$$

If the eccentric weights do not all rotate in the same plane, this condition alone is not sufficient to insure running balance. Thus consider a shaft carrying two eccentric weights W_1 and W_2 diametrically opposite and so arranged as to satisfy the condition for standing balance, $W_1 r_1 = W_2 r_2$, but not rotating in the same plane. When the shaft rotates, these eccentric weights generate centrifugal forces C_1 and C_2 equal in amount but not acting in the same line, and hence forming a couple which tends to revolve the shaft

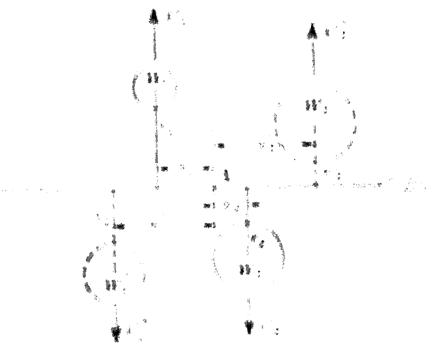


FIG. 279.

about an axis perpendicular to it, or make it wobble in its bearings. To counteract this effect an equal and opposite couple must be introduced by means of a pair of balance weights diametrically opposite the first pair, as indicated by the dotted lines in Fig. 279. Let the weights of these counterweights be denoted by W_1', W_2' and their eccentricities by r_1', r_2'. Then the condition for standing balance, $\sum Wr = 0$, becomes in this case

$$W_1 r_1 + W_2' r_2' - W_2 r_2 - W_1' r_1' = 0.$$

For running balance the moments of the centrifugal forces about any arbitrary point in the axis of rotation, say A, must be zero. Denoting these distances by x_1, x_2, etc., as indicated, this condition gives the relation

$$C_1 x_1 + C_2 x_2 - C_1' x_1' - C_2' x_2' = 0,$$

or, inserting the values of the centrifugal forces,

$$\frac{W_1 \omega^2 r_1 x_1}{g} + \frac{W_2 \omega^2 r_2 x_2}{g} - \frac{W_1' \omega^2 r_1' x_1'}{g} - \frac{W_2' \omega^2 r_2' x_2'}{g} = 0,$$

Cancelling out the common factor $\frac{\omega^2}{g}$, this reduces to

$$W_1r_1x_1 + W_2r_2x_2 - W_1'r_1'x_1' - W_2'r_2'x_2' = 0,$$

or simply $$\sum \mathbf{Wrx} = \mathbf{0}.$$

All cases of counterbalancing eccentric weights can be solved by applying one or both of the conditions $\sum Wr = 0$, $\sum Wrx = 0$.

For example, if a number of eccentric weights rotate in the same plane, only the first condition is necessary. In this case both standing and running balance may be secured by adding a single eccentric weight, rotating in the same plane as the given weights and such that the centrifugal force exerted by this counterweight shall equilibrate the centrifugal forces arising from the given set of eccentric weights.

If the eccentric weights rotate in different planes, each one may be treated separately and balanced by a pair of eccentric weights so arranged as to satisfy both the above conditions. If a pair of planes perpendicular to the axis of rotation is chosen arbitrarily at the outset, and each pair of counterweights added to balance one of the given eccentric weights is so chosen as to rotate in these planes, then when all the given weights have been balanced, half of the counterweights will lie in each of the assigned planes, and each set may then be replaced by a single counterweight, as explained above. A simple illustration of this method is given in Art. 159.

It is evident, therefore, that all cases of balancing eccentric weights may be solved by the application of the simple conditions

$$\sum \mathbf{Wr} = \mathbf{0},$$

$$\sum \mathbf{Wrx} = \mathbf{0},$$

and that, in general, not more than two counterweights need be added. It may also be noted that the first of these conditions is simply equivalent to saying that the center of gravity of all the weights taken collectively must lie in the axis of rotation, while the second condition is simply the principle of moments applied to the centrifugal forces arising from these weights.

155. Balancing Reciprocating Parts. In order that an engine may run smoothly at high speeds, the rotating and reciprocating parts must be carefully balanced. As regards the reciprocating parts it was shown in Art. [illegible] that for a connecting rod 7 cranks long the inertia thrust is $\frac{1}{7}$th greater at one end of the stroke and $\frac{1}{7}$th less at the other end than if the rod was infinitely long. Hence if the counterbalance for the reciprocating parts is made $\frac{1}{7}$th greater to allow for the obliquity of the rod at one end of the stroke, it will be $\frac{2}{7}$th too great at the other end. Therefore since more harm than good is done by trying to compensate for the obliquity of the connecting rod, it is customary to neglect this correction and proceed as though the rod was of infinite length.

Since the crank end of the connecting rod rotates while the other end reciprocates, it is customary to balance the crank end and half the plain part of the rod for rotation, and the other half of the rod and the remaining reciprocating parts for reciprocating motion. Since by reason of the obliquity of the connecting rod it is impossible to perfectly balance an engine, it is usual to balance about ⅔ of the reciprocating parts, this approximation being as close as warranted by the circumstances.

156. Single Crank Balancing. When no balance wheels are used, the moving parts may be balanced by placing weights in the plane of the crank webs, diametrically opposite the crank pin. An illustration of this form of balancing is shown in Fig. 280. The size of these balance weights may be determined as follows:

Let W_r = weight of rotating parts
(= crank pin + crank webs + large end of connecting rod + one half plain part of rod),

W_p = weight of reciprocating parts
(= piston + piston rod + cross head + small end of connecting rod + half plain part of rod + attachments such as valves, etc.),

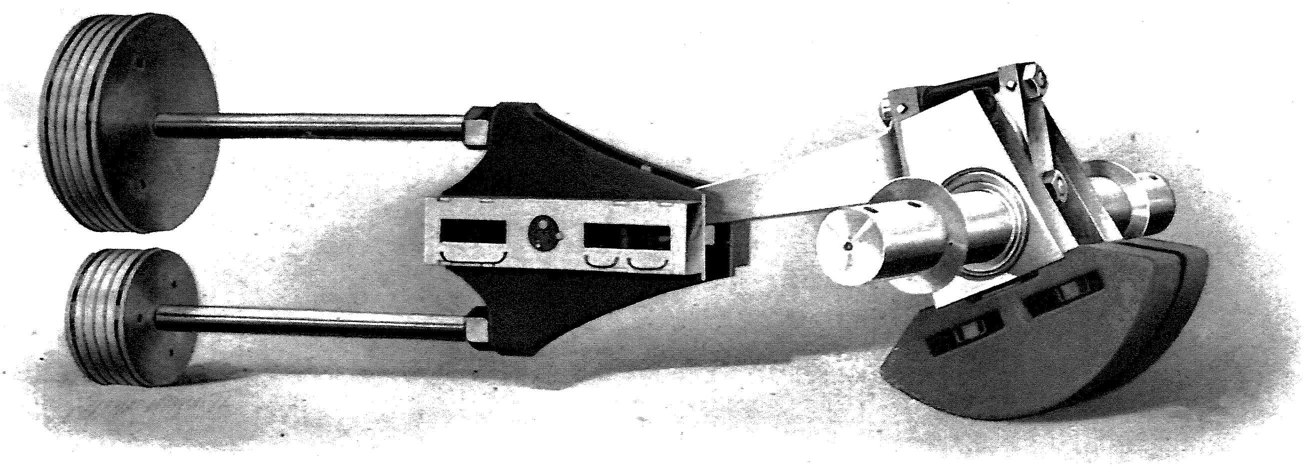


FIG. 280

155. Balancing Reciprocating Parts. In order that an engine may run steadily at high speeds, the rotating and reciprocating parts must both be carefully balanced. As regards the reciprocating parts, it was shown in Art. 150 that for a connecting rod q cranks long the inertia thrust is $\frac{1}{q}$th greater at one end of the stroke and $\frac{1}{q}$th less at the other end than if the rod was infinitely long. Hence if the counterbalance for the reciprocating parts is made $\frac{1}{q}$th greater to allow for the obliquity of the rod at one end of the stroke, it will be $\frac{2}{q}$th too great at the other end. Therefore since more harm than good is done by trying to compensate for the obliquity of the connecting rod, it is customary to neglect this correction and proceed as though the rod was of infinite length.

Since the crank end of the connecting rod rotates while the other end reciprocates, it is customary to balance the crank end and half the plain part of the rod for rotation, and the other half of the rod and the remaining reciprocating parts for reciprocating motion. Since by reason of the obliquity of the connecting rod it is impossible to perfectly balance an engine, it is usual to balance about $\frac{2}{3}$ of the reciprocating parts, this approximation being as close as warranted by the circumstances.

156. Single Crank Balancing. — When no balance wheels are used, the moving parts may be balanced by placing weights in the planes of the crank webs, diametrically opposite the crank pin. An illustration of this form of balancing is shown in Fig. 280. The size of these balance weights may be determined as follows:

Let W_r = weight of rotating parts
(= crank pin + crank webs + large end of connecting rod + one half plain part of rod),

W_e = weight of reciprocating parts
(= piston + piston rod + cross head + small end of connecting rod + half plain part of rod + attachments such as valves, etc.),

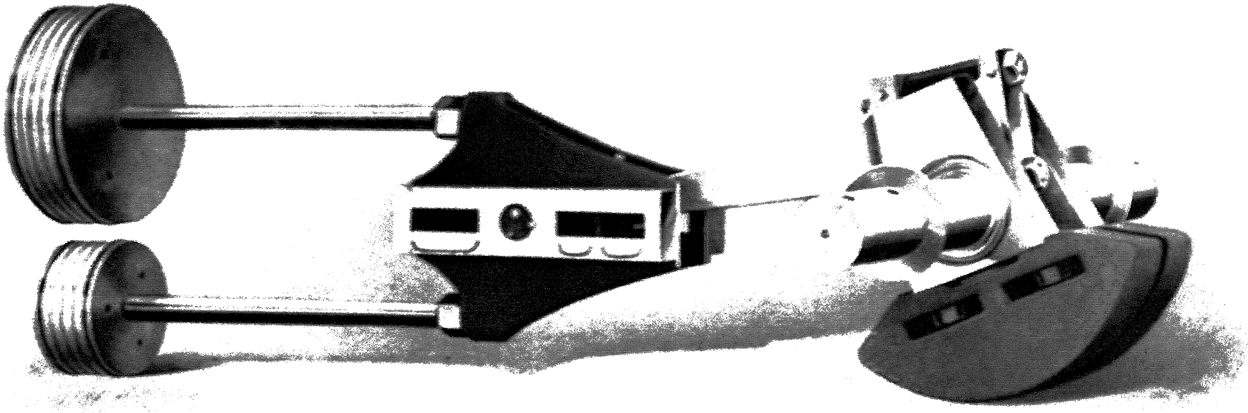


FIG. 280

W_B = weight of each counterweight in plane of web,
r = radius of crank,
r_B = distance from center of gravity of counterbalance to axis of shaft.

Then, if we follow the ordinary rule of balancing two thirds of the reciprocating parts, we have, by equating the centrifugal forces arising from these and the counterweights,

$$2\,W_B r_B = (W_c + \tfrac{2}{3}\,W_r)r,$$

whence

$$W_B = \frac{(W_c + \frac{2}{3}\,W_r)r}{2\,r_B}.$$

Since the two balance weights are symmetrically placed at equal distances from the center of the crank pin, it is not necessary in this case to apply the condition $\sum Wrx = 0$. In fact, taking moments about the center of the crank this condition would become $W_B r_B x + W_B r_B(-x) = 0$, which is simply an identity.

157. Double Crank Balancing. When there are two cranks and consequently two sets of reciprocating parts to balance, they may each be balanced separately as shown in Fig. 281, which rep-

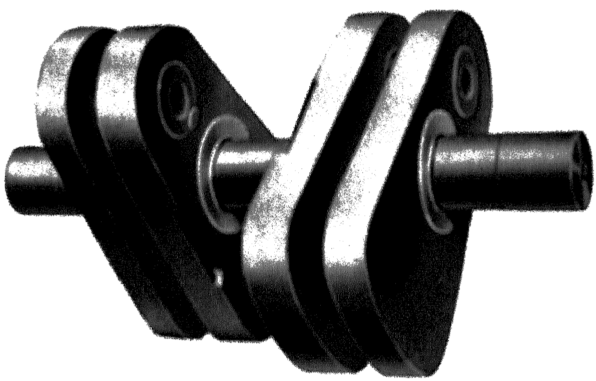


FIG. 281

resents an axle designed by Mr. Drummond for the London and Southwestern Railway; or they may both be balanced by a single counterweight at an angle of 45° to each of the cranks as shown in Fig. 282, which represents another type of locomotive axle.

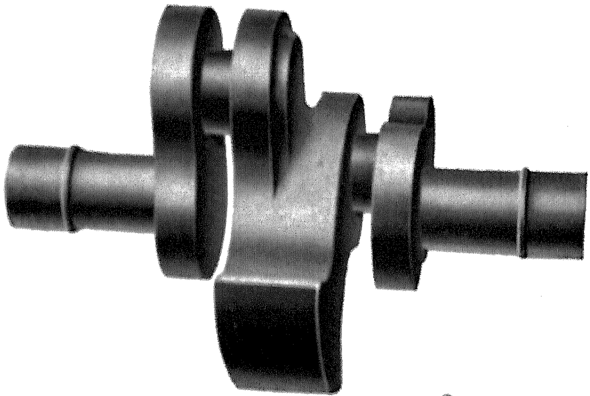


FIG. 282

Another type of crank balancing is shown in Fig. 283, which represents a cross section of an angle compound engine. In this case there is only one crank, but two sets of reciprocating

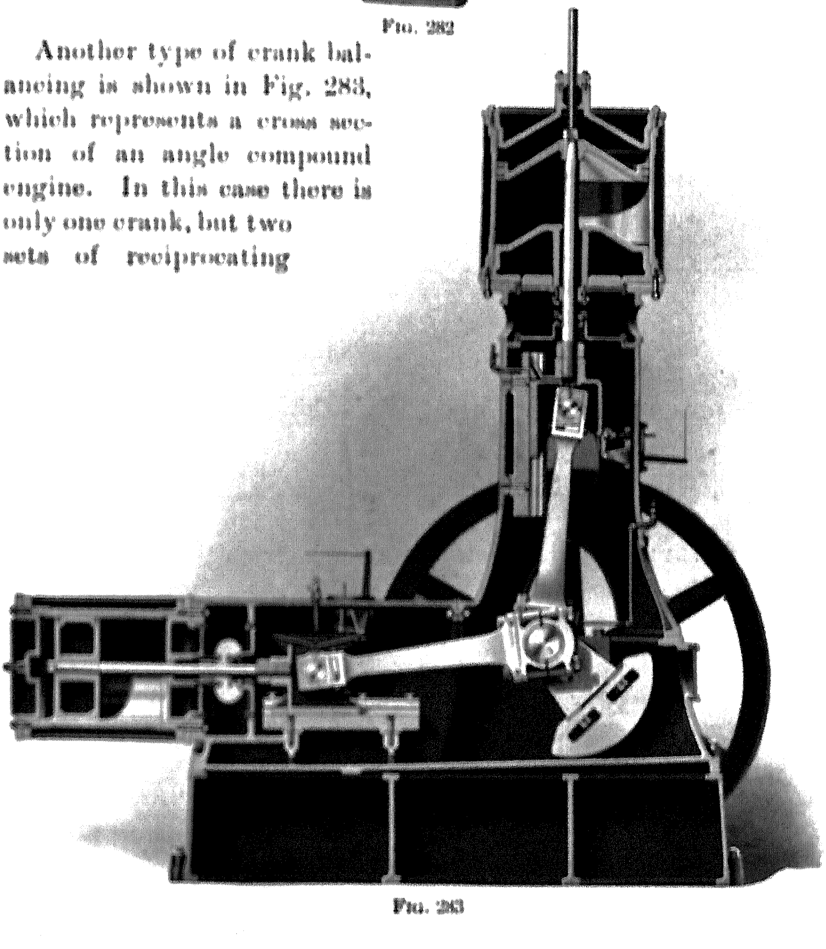


FIG. 283

parts, the center line of the counterweight bisecting the angle between them. In designing this type of engine the two sets of reciprocating parts are made as nearly as possible of the same weight. This is accomplished by dishing the piston in the larger cylinder to make it lighter, and increasing the weight of the smaller piston as much as may be necessary to make the two sets of reciprocating parts of equal weight. The counterweight is then made of sufficient size to balance the inertia thrust of the horizontal reciprocating parts when passing the horizontal centers. The same counterweight will then approximately balance the vertical thrust of the vertical parts when passing the vertical centers, and at all intermediate positions the sum of these thrusts will also be approximately balanced.

158. Counterweighting Balance Wheels. In some types of engine the inertia thrust of the rotating and reciprocating parts is balanced by adding counterweights to the flywheel or pulley attached to the shaft. Evidently two balance wheels must be used, for if only one were used a couple would be produced which would tend to cause the shaft to wobble in its bearings, as explained in Art. 152. The objection to this method of balancing is that as the balance wheels are necessarily at some distance from the crank, the shaft is subjected to a considerable bending strain.

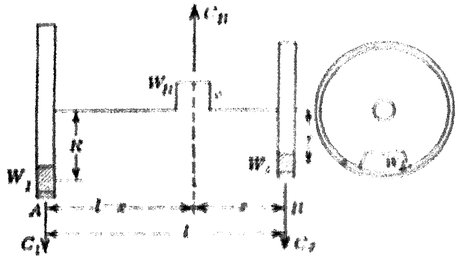


FIG. 284

Consider first a single crank with two balance wheels, as shown in Fig. 284.

Let W_1 = weight of counterbalance on larger wheel,

W_2 = same for smaller wheel,

R, r = distances from center of gravity of W_1, W_2 to axis of shaft,

c = radius of crank,

l = distance between centers of counterweights measured parallel to shaft,

W_R = weight to be balanced = $\frac{2}{3} W_{recip} + W_{rot}$,

C_1, C_2, C_R = centrifugal forces developed by W_1, W_2, W_R.

For equilibrium the sum of the moments of the centrifugal forces about any given point must be zero. Thus from

$$\sum \text{moments about } A = 0$$

we have
$$C_2' l = C_B(l - x), \text{ or}$$
$$W_2 r l = W_B e(l - x);$$

and similarly from

$$\sum \text{moments about } B = 0$$

we have
$$W_1 R l = W_B e x.$$

Consequently
$$W_1 = W_B \frac{e}{l} \cdot \frac{x}{R} \text{ and } W_2 = \frac{W_B e}{l} \cdot \frac{l - x}{r}.$$

If the two balance wheels are of the same size, then $R = r$ and

$$W_1 + W_2 = \frac{W_B e}{l} \cdot \frac{l}{r} = \frac{W_B e}{r}.$$

159. Inside Two-cylinder Engine.—As another example of counterweighting balance wheels, consider a two-cylinder engine with cranks inside the bearings and balance wheels outside, as shown in perspective in Fig. 285. An example of this type is the fire engine shown in Fig. 286.

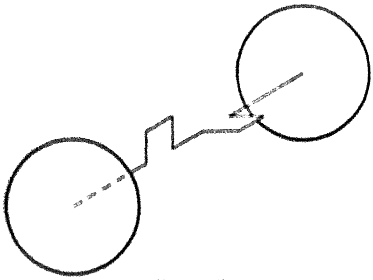
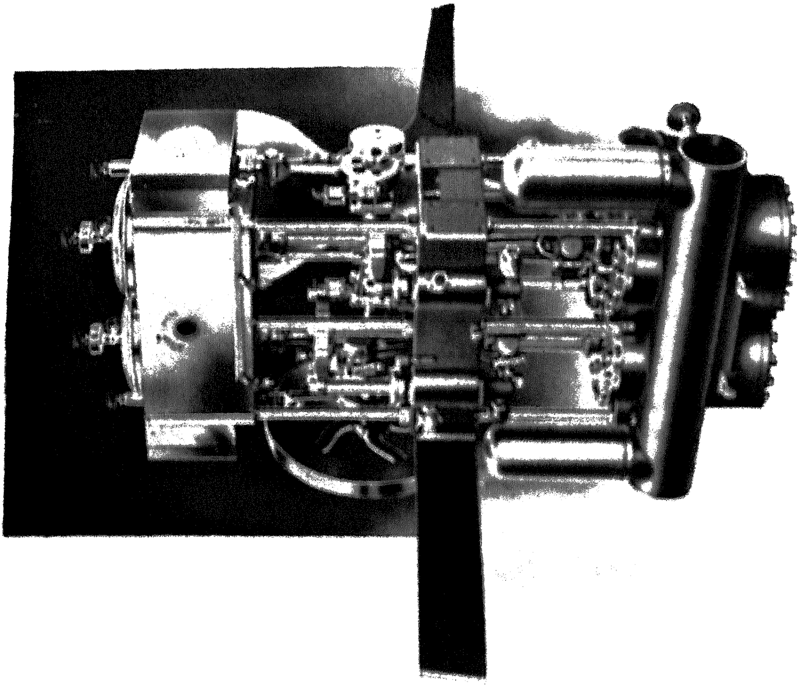
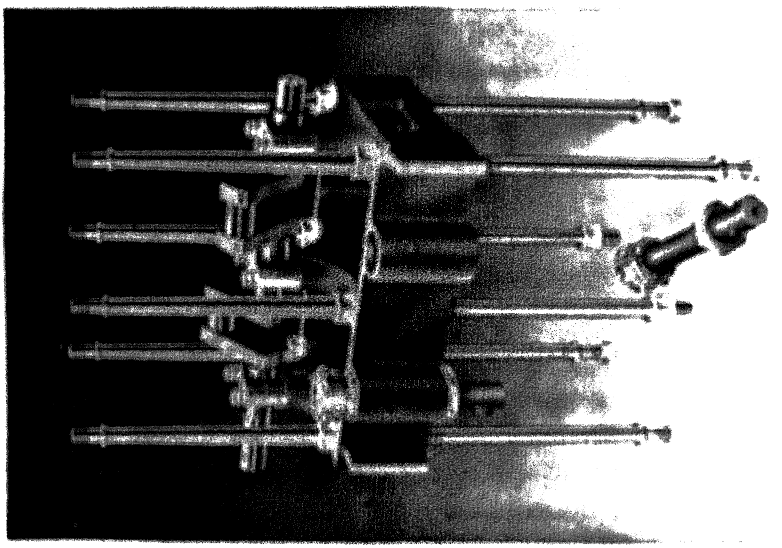


FIG. 285

The simplest method of procedure in this case is to balance each crank separately as explained in the preceding article and then combine the results. Let the notation for weights and dimensions be as shown in Fig. 287 with cranks 90° apart, and begin by balancing the crank lettered A. Then if the two balance wheels are of the same size and r_B denotes the radius of the counterweight on each wheel, the total amount of the balance weight to be distributed between the wheels is

$$W_A = (W_c + \tfrac{2}{3} W_r)\frac{r}{r_B},$$





where W_r and W_c denote the weights of the rotating and reciprocating parts, respectively, attached to crank A. The fraction of

Fig. 287

this to be placed on each wheel is determined by taking moments about the center of the opposite wheel seat. Thus taking moments about the right end D, we have

$$W_1 l = W_s\left(c + \frac{l - c}{2}\right) = W_s \frac{l + c}{2},$$

whence

$$W_1 = W_s \frac{l + c}{2l}.$$

Similarly, taking moments about the left end E,

$$W_2 = W_s \frac{l - c}{2l}.$$

These weights, W_1 and W_2, placed on the balance wheels at the radius r_0 will balance the left-hand crank and parts. The other crank is then balanced in the same manner, with a similar result since the arrangement is symmetrical. Hence, on each wheel there will be one large weight W_1 diametrically opposite the near crank and a smaller weight W_2 diametrically opposite the far crank, as shown in Fig. 288. These two counterweights may then be combined by the parallelogram law into a single counterweight W, of amount

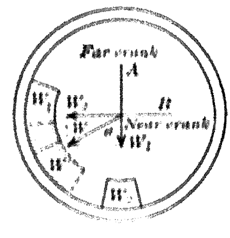


Fig. 288

$$W = \sqrt{W_1^2 + W_2^2} = \frac{W_s}{l}\sqrt{\frac{l^2 + c^2}{2}}.$$

The angle θ which the radius of this counterweight makes with the near crank is given by

$$\tan\theta = \frac{W_2}{W_1} = \frac{l - c}{l + c}.$$

These two equations therefore completely determine the position and amount of the counterweights.

Instead of adding the weight W to the rim of the balance wheel, an equal weight may be cored out of the rim at a point diametrically opposite. The balancing effect will be the same and the appearance is neater, as it makes the balance wheels appear perfectly symmetrical. This device is used in balancing the fire engine shown in Fig. 286.

PROBLEM

418. In the fire engine shown in Fig. 289 the weights and dimensions for balancing are as given below. Determine the size and position of the cored segment in the rim of each balance wheel, assuming that all the rotating, and two thirds of the reciprocating, parts are balanced.

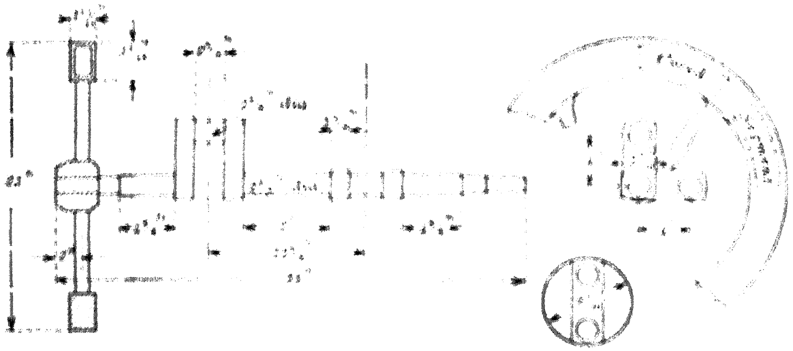


Fig. 289. Ends of Crank Cheeks turned to 8⅛"

Reciprocating Elements	Weight in lb.	Rotating Elements	Weight in lb.
1 steam piston head	21	1 crank shaft	56¼
1 steam piston rod and top yoke	18¾	1 flywheel	10⅞
4 yoke rods	26	2 eccentrics	11
1 pump piston rod and bottom yoke	19¼	*Semi-reciprocating and Rotative*	
1 pump piston head	13¼	1 connecting rod	20¼
1 feed pump ram	6	length centers 16 in., center of gravity 8¾ in. from center of head	
Dimensions		1 eccentric strap	9
See Fig. 289.		1 valve rod	5¼
		1 slide valve	5¼

SOLUTION. Assuming the weight of steel to be 490 lb./ft.³, the weight of each crank web is found to be 6.567 lb. and the weight of the crank pin to be 2.640 lb. Therefore, neglecting the weight of eccentrics and slide valve, and assuming that the rotating parts are concentrated at the crank pin, we have

$$W_{recip} = 10.69 \times 10.375 = 110.875 \text{ lb.},$$

$$W_{rot} = (2.640 - 2(0.0607)) \times 10.375 = 26.129 \text{ lb.},$$

$$W_c = \frac{r}{r_c}\left(W_{rot} + \tfrac{2}{3}W_{recip}\right) = \tfrac{4}{10}(100) = 40 \text{ lb.}$$

Since the distance between centers of counterweights is $l = 34$ in. and the distance between centers of cranks is $c = 12.5$ in., the total counterweight W on each wheel is

$$W = \frac{W_c}{l}\sqrt{\frac{l^2 + c^2}{2}} = 30.3 \text{ lb.};$$

and the angle its radius makes with the near crank is given by

$$\tan\theta = \frac{l - c}{l + c} = \frac{21.5}{46.5},$$

whence $\theta = 24° 49'$.

160. Outside Cylinder Locomotive. — As a typical case of locomotive balancing consider the six-wheeled outside cylinder engine shown in Fig. 291. Since the connecting rod is coupled to the middle driver, the weight to be balanced will be greater for this driver than for either of the others. Hence starting with the middle driver, or wheel No. 2 in Fig. 290, let

FIG. 290

W_a = weight of rotating parts, including side rod from A to B,

W_r = weight of reciprocating parts

= (piston + piston rod + cross head + small end of connecting rod + $\frac{1}{3}$ plain part of rod),

W_B = weight to be counterbalanced = $W_a + \frac{2}{3}W_r$,

c = crank radius,

r_2 = radius of counterweight on wheel No. 2.

Then $W_2 r_2 = W_B c$, whence

$$W_2 = W_B \frac{c}{r_2}.$$

For wheels Nos. 1 and 3 the parts to be counterbalanced consist only of the crank pins and the remaining portions of the side rod.

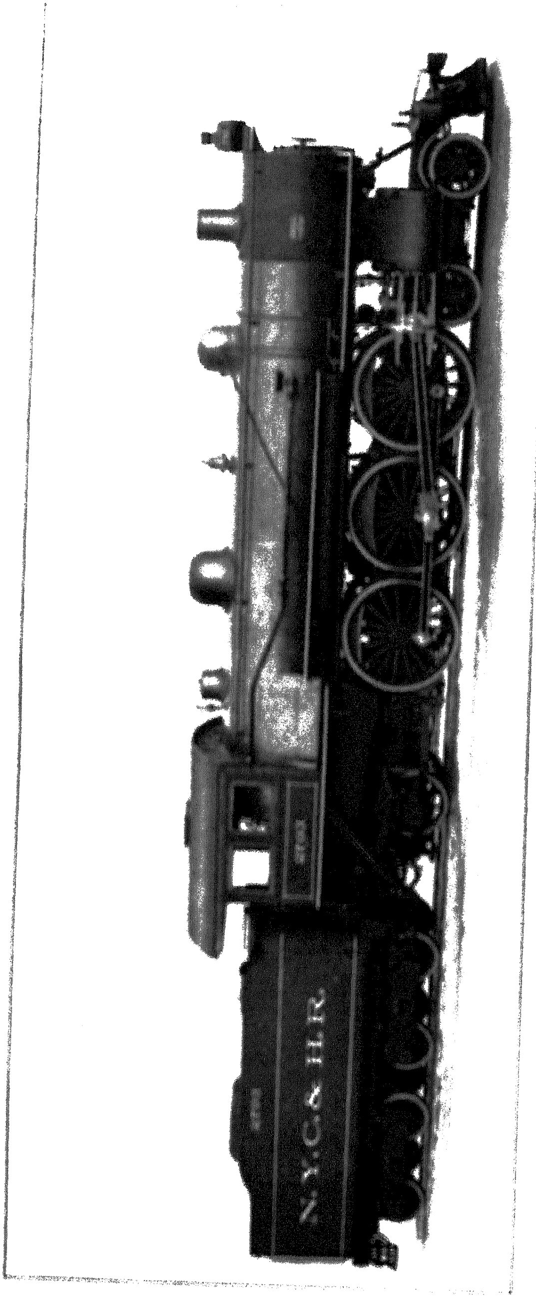


FIG. [illegible] — [illegible]

Denoting this by W_d' for each wheel, we have

$$W_1 = W_3 = W_d' \frac{r}{r_1}.$$

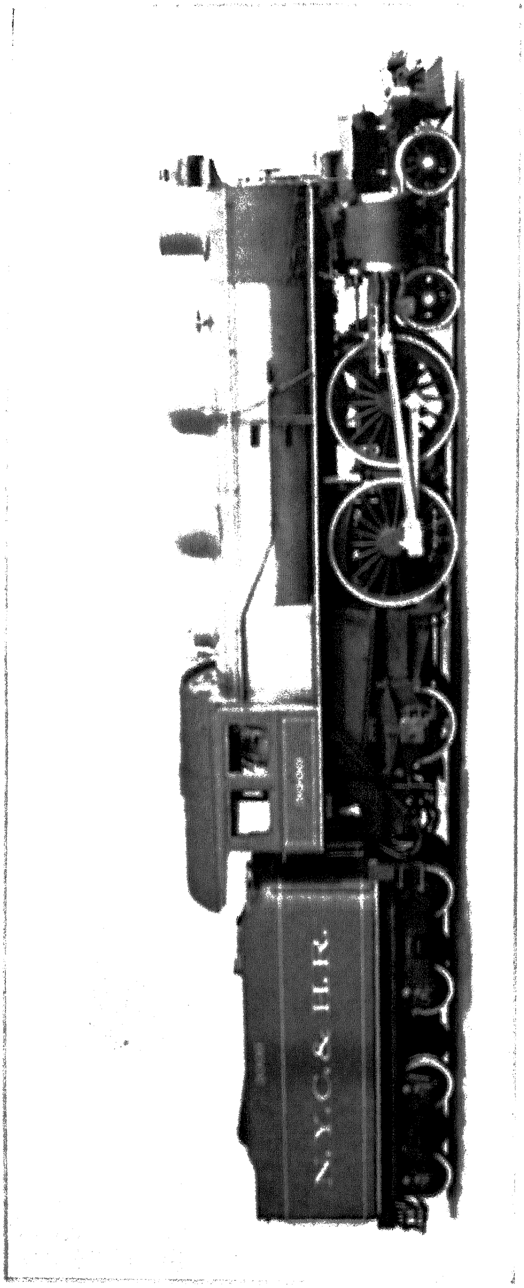
The relative size of the counterweights is shown in Fig. 291.

161. Balanced Compound Locomotive. In balancing an ordinary locomotive the counterweight is designed to balance the horizontal reciprocating parts, as explained in the preceding article. Under this arrangement, however, the engine is overbalanced vertically, the result being to produce a hammer blow on the rails at each revolution. For an ordinary ten-wheel locomotive weighing 165,000 lb., with 123,000 lb. on drivers, cylinders 20 in. × 26 in., driving wheels 69 in., the static and dynamic weights on the drivers are as follows:

Speed in Miles per Hour	Static Weight	Dynamic Weight	Increase per Pair of Wheels
50	123,000	140,500	5,840
60	123,000	148,000	8,340
70	123,000	157,200	11,400

As the limitation of weight on the drivers of a locomotive which the rails and roadbed will safely withstand is the dynamic and not the static load, it follows that a more powerful locomotive can be built if more perfect balancing can be secured.

This demand has resulted in the four-cylinder balanced compound locomotive, illustrated in Fig. 292. Four cylinders are used in this type, two low-pressure cylinders outside the frame as in an ordinary locomotive, and two high-pressure cylinders inside the frame. The inside cranks for the high-pressure cylinders are 90° apart, as shown in Fig. 293, and the cranks for the low-pressure cylinders are diametrically opposite those for the high. Thus one inside piston and its attachments are moving forward, while the corresponding outside piston is moving backward. Hence, by making the weights of these reciprocating parts equal, they are made to exactly balance each other without the addition of counterweights on the wheels.


N.Y.C.&H.R.R.

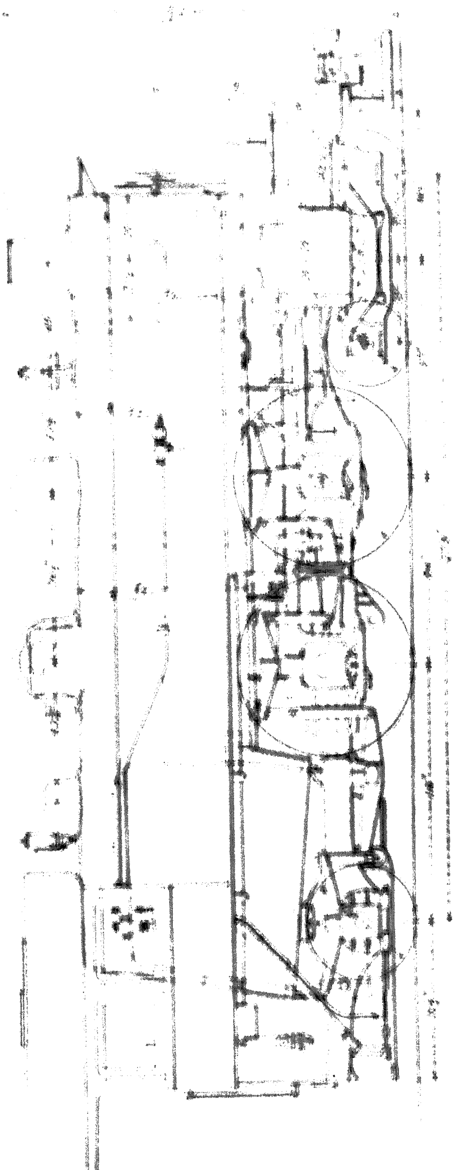


Fig. [illegible]a. — *Side elevation.*

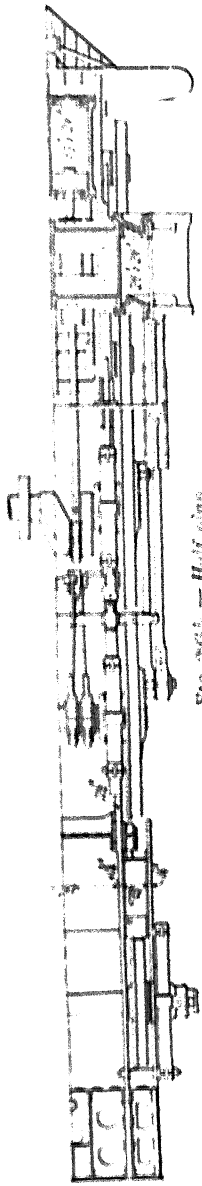


Fig. [illegible]b. — *Half plan.*

Cole four-cylinder balanced compound passenger locomotive, Atlantic type.

The method of making the reciprocating parts for the high and low pressure cylinders of equal weight is shown in Fig. 294, the low-pressure piston being dished to make it light, while the high-pressure piston is made solid. A similar arrangement is also shown in Fig. 283.

Since the reciprocating parts are thus made to balance each other, it is only necessary to put counterweights on the wheels sufficient to balance the rotating parts. By comparing Figs. 290

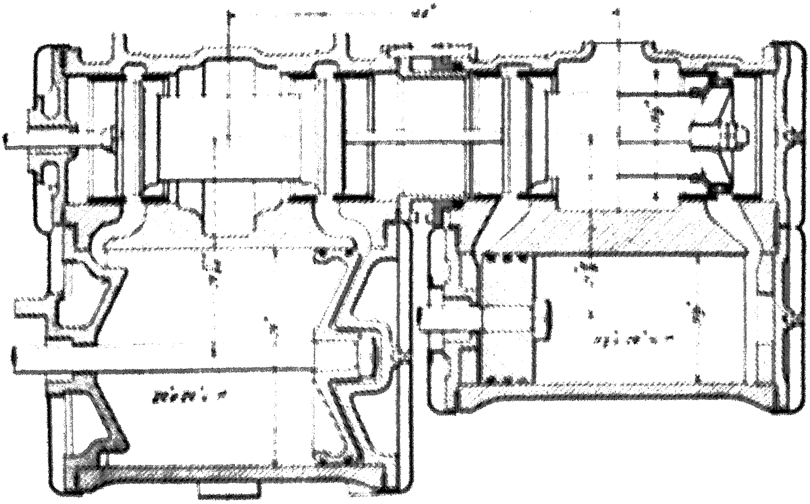


Fig. 294

and 292, it will be seen that the counterweights are considerably smaller in the four-cylinder balanced compound than in the ordinary type of locomotive. A much more perfect balance is also secured in the former, and the hammer blow on the rail is practically eliminated.

162. Conservation of Momentum In Art. 48, Chapter II, there was given an elementary demonstration of the principle of the conservation of linear momentum, and in Art. 53 of the same chapter this principle was extended to include angular momentum. The principle of the conservation of energy, derived in Art. 130, Chapter VI, is perhaps usually considered as the most important principle of mechanics, but in view of recent developments in commercial applications of mechanics, the principle of

the conservation of angular momentum is proving to be of much greater practical utility.

Energy is a scalar quantity, that is to say, a numerical magnitude without direction, and can therefore be specified by a single condition. Momentum, however, is a vector quantity, having direction as well as magnitude, and therefore requires three conditions to specify it. In consequence of this, the conservation of momentum is a more powerful principle than the conservation of energy.

The main reason for the importance of this principle is that most of the modern applications of mechanics are concerned chiefly with rotation. The inertia forces called into play by rotation afford a means of control which is entirely lacking in static mechanisms, and likewise in motions of translation, which, as shown by d'Alembert's principle, are closely related to statics. Whenever rotation occurs, angular momentum is generated, and by applying the principle of the conservation of angular momentum, the kinetic reactions may be calculated and the problem of motion completely solved. Other methods, such as Hamilton's principle and Euler's equations, might be used to solve such problems, but afford a much less obvious and direct method of treatment than the principle of the conservation of momentum. The method of applying this principle is illustrated in what follows by several of the more recent applications of gyroscopic control.

163. Simple Illustrations of the Conservation of Angular Momentum. As shown in Art. 53, Chapter I, the expression for the angular momentum in the case of a body rotating about a fixed axis is

$$H = I\omega,$$

where I denotes its moment of inertia and ω its angular velocity with respect to this axis.

To illustrate the principle

$$H = \text{constant},$$

consider a shaft carrying a fixed pulley keyed to the shaft and a loose pulley or idler, and suppose that when the shaft and fixed pulley are rotating at a speed ω the loose pulley is suddenly coupled up by throwing in a clutch. Then by the principle of

the conservation of angular momentum the total angular momentum of the two pulleys and shaft must be the same as that of the single fixed pulley and shaft before the clutch was thrown in. Thus if I_1 denotes the moment of inertia of the fixed pulley and shaft, I_2 the moment of inertia of the loose pulley, and ω' their speed when coupled together, the condition H = constant becomes in this case $I_1\omega = (I_1 + I_2)\omega'$, whence

$$\omega' = \frac{I_1}{I_1 + I_2}\omega.$$

As a further illustration, consider the piece of apparatus shown in Fig. 295. This consists of a horizontal arm rotating about a fixed vertical axis CD and carrying two equal weights A and B, which are free to slide along the horizontal arm except as they are constrained by the strings attached to them and meeting in C. Let the weight of each slider, A or B, be denoted by W, and let r denote the distance of its center of gravity from the axis of rotation CD.

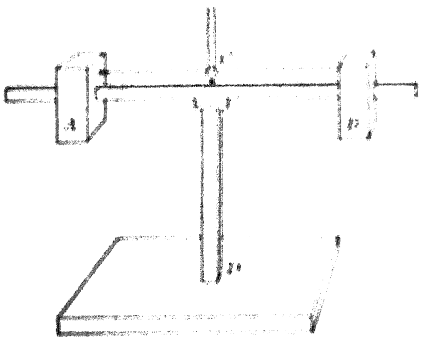


FIG. 295

Then suppose that when the arm with its weights is revolving at an angular velocity ω the strings are suddenly let out an amount l, so that the weights fly out to a distance $l + r$ from the axis. Then, applying the principle of the conservation of angular momentum, if the new angular velocity is denoted by ω_1, we have

$$\frac{Wr^2\omega}{g} = \frac{W(l + r)^2\omega_1}{g},$$

whence

$$\omega_1 = \left(\frac{r}{l + r}\right)^2\omega.$$

The angular velocity is therefore decreased by allowing the weights to move out from the axis. Similarly, their angular velocity increases as they approach the axis.

Another familiar example of this principle is that of a stone swung at the end of a string which is allowed to wind up on the finger. As the string winds up the speed of the stone increases, and vice versa.

164 Vector Representation of Angular Momentum. — In Art. 24, Chapter I, it was shown that all quantities having direction as well as magnitude may be represented by a line of given length and direction, called a vector. In the case of angular momentum this method of vector representation affords a means whereby problems which would otherwise be quite complicated and more or less obscure may be fully treated in a clear and elementary manner.

For example, consider a homogeneous cylinder, such as a disk flywheel or pulley, of radius r and rotating about its geometric axis with angular velocity ω. Since the radius of gyration of a circular cylinder is the same as the polar radius of gyration of its circular cross section, namely $k = \frac{r}{\sqrt{2}}$, if W denotes the weight of the cylinder its moment of inertia with respect to its geometric axis is $I = \frac{Wr^2}{2g}$. The angular momentum of the cylinder may therefore be represented by a vector lying in the axis of the cylinder, as shown in Fig. 290, whose length represents to any given scale the numerical value of the angular momentum, namely $\frac{Wr^2\omega}{2g}$, and whose direction is that in which a right handed screw would advance along the axis if turned in the direction of rotation.

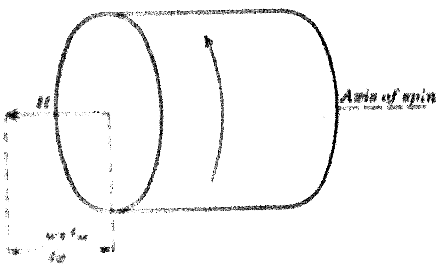


Fig. 290

In the first example considered in the preceding article, the angular momentum of the fixed pulley and shaft is represented by the vector $I_1\omega$. After the loose pulley is connected up, the angular momentum of the fixed pulley and shaft becomes $I_1\omega'$, and that of the loose pulley is $I_2\omega'$, and the principle H = constant says that the sum

of these two vectors must be equal to the original vector $I_1\omega$ (Fig. 297).

In the second example given in the preceding article the angular momentum is $2I\omega$ in any position of the weights. This, however, is the product of two quantities, I and ω, and as one increases

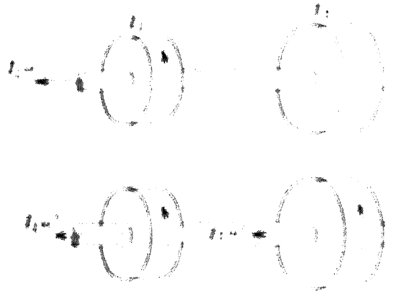


FIG. 297.

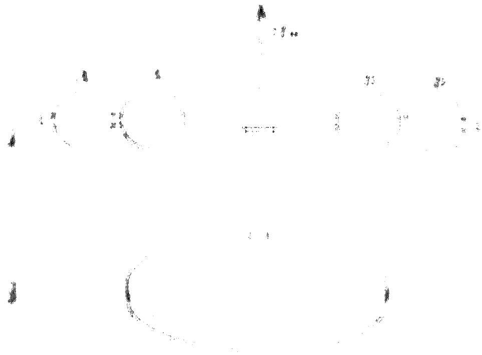


FIG. 298.

the other decreases so as to keep their product constant (Fig. 298). For a more extended discussion of this piece of apparatus see Art. 129, Chapter VI.

In all the preceding examples the direction of the vector momentum remained fixed although its length varied. A more important case is that in which the direction of the momentum changes as well as its length. Let AB, Fig. 299, represent the vector momentum at any given instant, and after an interval of time Δt let it be represented by AC. Then BC represents the vector change in momentum in the interval Δt. Let this be denoted by ΔH. Then since

FIG. 299.

$$\frac{\Delta H}{\Delta t} = M, \text{ or } \Delta H = M\,\Delta t,$$

it is evident that an impulsive couple $M\Delta t$ is required to change the angular momentum H by an amount ΔH.

To illustrate the effect of a change in the direction of the angular momentum, consider a flywheel or pulley as keyed on a shaft

that its axis does not coincide with the axis of the shaft. Let β denote the angle between the two axes (Fig. 300). Then when the shaft has revolved through an angle θ, the vector momentum will have changed in direction by an angle ϕ such that

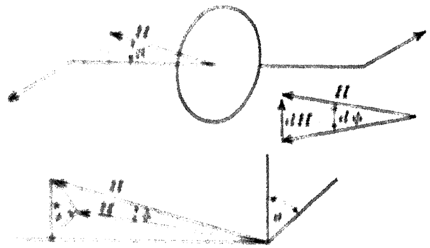


Fig. 300

$$d\phi = d\theta \sin \beta.$$

Now the change in momentum dH gives rise to (or is produced by) an impulsive couple $M\,dt$. Also from the vector triangle in Fig. 300 we have $dH = H\,d\phi$. Therefore

$$dH = M\,dt = H\,d\phi = H\,d\theta \sin \beta,$$

whence

$$M = H\frac{d\theta}{dt}\sin \beta = H\omega \sin \beta.$$

Let W denote the weight of the wheel and k its radius of gyration. Then since $I = \dfrac{Wk^2}{g}$ and $H = I\omega$, the expression for the moment becomes

$$M = I\omega^2 \sin \beta = \frac{Wk^2\omega^2 \sin \beta}{g}.$$

Now let l denote the distance between centers of bearings and R the kinetic reaction on each. Then

$$Rl = M = \frac{Wk^2\omega^2 \sin \beta}{g},$$

and consequently

$$R = \frac{Wk^2\omega^2 \sin \beta}{gl}.$$

The static pressure on the bearings therefore increases and decreases by this amount R, going through its entire range once in each revolution. Since the kinetic reactions R act in opposite directions at the two ends of the shaft, the tendency is for one end of the shaft to lift out of the bearing, while at the same time the downward pressure at the other end is increased.

PROBLEM

419. A flywheel weighing [illegible] lb. and having a radius of gyration of 2 ft. is keyed on a shaft so that its plane makes an angle of 1° with a plane perpendicular to the axis of the shaft. Find how much this obliquity increases the reactions on the bearings when revolving at 150 r.p.m.

165. Gyroscopic Action. — The kinetic reactions developed by a couple so acting as to produce a change in angular momentum, as illustrated in the preceding article, is an example of what is known as gyroscopic action.

FIG. 301

The law of gyroscopic motion may be illustrated very simply by means of a hollow top, so constructed that its center of gravity shall coincide with the point of support, as shown in Fig. 301. Since the point of support is at the center of gravity, the top is in neutral equilibrium, and will spin in any position without any tendency to alter the direction of its axis. If while the top is spinning a string is looped over the end of its axis and pulled to one side, the point A will move off in a direction at right angles to the pull, as indicated by the arrow AH in Fig. 301. Such motion at right angles to the direction of the constraint is characteristic of gyroscopic action.

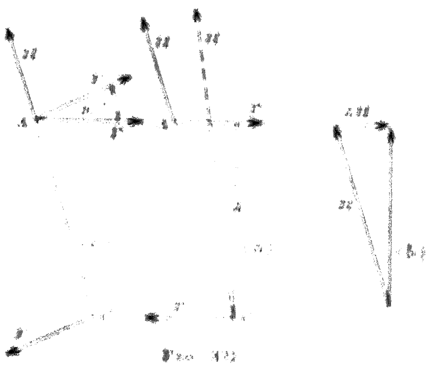


FIG. 302

The explanation of this phenomenon is that the constraint P tends to produce a change in the direction of the vector angular momentum. Thus in Fig. 302 (a) let AC represent the axis of the top when spinning freely in any position. Then for a right-handed spin as shown, the vector momentum H is directed outward along the axis. The constraint P, however,

produces a couple Pd about the point of support C which tends to pull the axis of the top inward. The vector momentum H is thereby also pulled in toward the vertical, producing a vector change dH perpendicular to H, as shown in Fig. 302 (b). Now $dH = Mdt$ (see preceding articles, where Mdt is an impulsive couple having the same direction as dH. If, however, the couple Mdt has the direction dH, the forces of this couple will lie in a plane at right angles to this direction. Thus if F, F denote the forces of this couple, they will act in a direction at right angles to the plane in which the vectors H, P and dH lie. One of the forces F of the couple acts at the point of support C, and the other acts at the other point of constraint A, and at right angles to the constraint as just proved. It is due to this unbalanced force F, or couple Fd, that the point A moves off at right angles to the direction of the constraint.

Another simple piece of apparatus which may be used to illustrate the same phenomenon is the ordinary gyroscope hung in gimbals (Fig. 303). This consists of a small flywheel, usually made of lead which when set in rapid rotation generates the angular momentum required for the experiment. To illustrate its action, suppose that the flywheel is given a rapid spin and one end of the axis

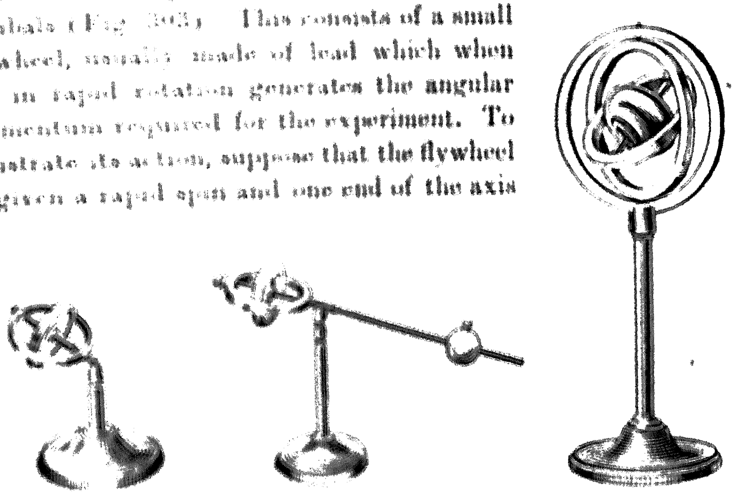


FIG. 303

is then laid in a loop at the end of a string hanging vertically or in a cup at the top of a short post as shown in Fig. 303, so that the axis of the top is horizontal. If the top is then released its axis will remain horizontal and the string vertical, but the top will revolve about the string as a vertical axis without showing any tendency to fall.

The explanation of this is substantially the same as above. When the axis of the spinning gyroscope is placed horizontal and released, the weight of the top tends at first to cause the end A to sink (Fig. 304). That is to say, there is at first a tendency for AB to revolve downward about B. The direction of the angular momentum H is thereby altered, the vector change in momentum dH being perpendicular to H, or in the present case directed vertically downward.

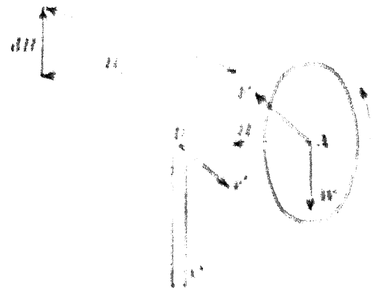


Fig. 304

Since dH is vertical, it is equivalent to a horizontal couple F, F which tends to rotate the axis AB about the vertical BC, one force being applied at the point of support B and the other acting at the outer end A (Fig. 304). After the gyroscope has begun to rotate horizontally about the vertical axis BC, or **precess** as it is called, the vector momentum H also moves horizontally. Consequently the vector change in momentum dH now becomes horizontal, that is, tangential to the circle having AB as radius, as shown in Fig. 305. Hence dH is now equivalent to a couple in a vertical plane the forces of which are the weight of the gyroscope W at A, and the equal and opposite reaction W at B. The horizontal couple therefore disappears as soon as precession is set up, and the rate of precession, or vector change in momentum $dH = Mdt$, is such as to balance the applied couple $M = W \times AB$.

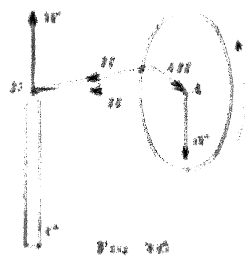


Fig. 305

As a numerical illustration of the preceding, suppose that the length of the arm AB is $l = 6$ in., the angular velocity of the gyroscope is $\omega = 100$ revolutions per second and its radius of gyration is $k = 3$ in. Then

$$H = I\omega = \frac{Wk^2\omega}{g}.$$

Also if Ω denotes the angular velocity of precession about BC, the

change dH in the time dt is

$$dH = H\Omega\, dt = \Omega \frac{Wk^2\omega}{g} dt = M\, dt = Wh\, dt.$$

Therefore
$$Wh = \Omega \frac{Wk^2\omega}{g},$$

and hence
$$\Omega = \frac{hg}{\omega k^2}.$$

Inserting the given numerical values in this expression, it is found that the angular velocity of precession is

$$\Omega = \frac{\frac{1}{2} \times 32.2}{(\frac{1}{2})^2\, 100 \times 2\pi} = 0.41 \text{ revolution per second.}$$

The period of the precession, or time required for a complete revolution, is therefore

$$P = \frac{2\pi}{0.41} = 15.3 \text{ sec.}$$

As the frictional resistance decreases the angular velocity of spin ω, the precessional velocity Ω will increase.

The gyroscope used by the author for class demonstration consists of a bicycle wheel running on ball bearings, with the tire removed and replaced by a section of heavy lead pipe. This wheel is mounted on a horizontal arm which is supported on a vertical standard by a universal joint so that it is free to turn in any direction. This arm carries two counterweights by means of which it may be balanced or unbalanced to any desired extent (Fig. 306).

If the arm is exactly balanced and the wheel set in rotation, of course no change in direction occurs. If, however, the arm is unbalanced and the wheel given a spin and then released, the arm will begin to precess horizontally about the vertical axis of the standard. The explanation of this is precisely the same as just given for the gyroscopic top. The effect, however, may be much better observed since it is only necessary to take hold of the end of the arm to feel the extent of the kinetic reactions. Suppose, for instance, that the wheel is given a right-handed spin so that the vector momentum will be directed outward from the

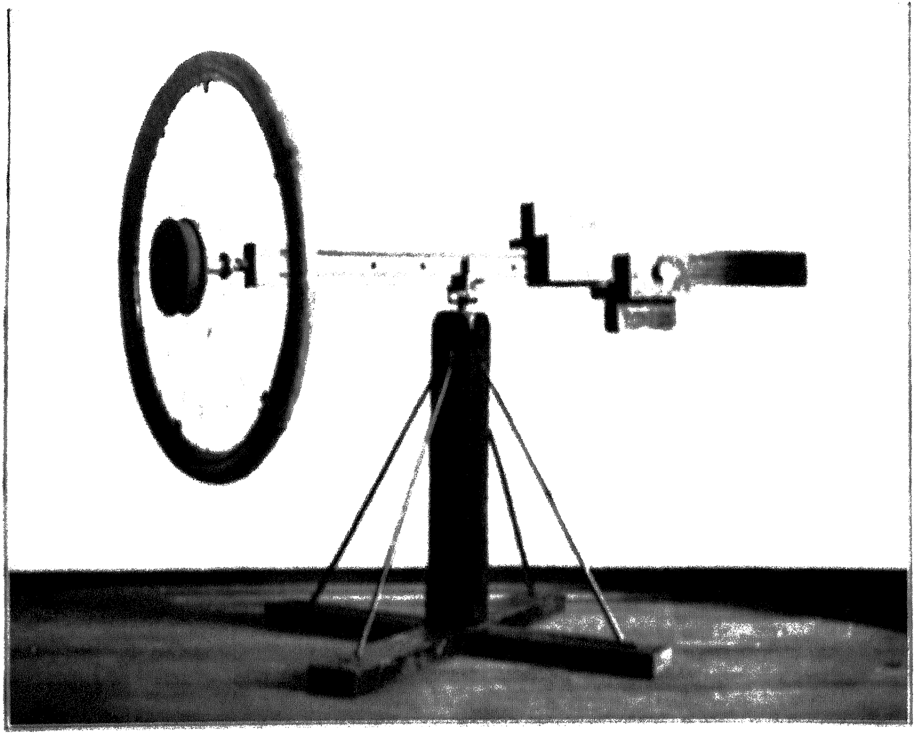


Fig. 306.

center. Also let the counterweights be so placed that they will tend to fall and the wheel to rise. The arm will then begin to precess horizontally in the direction indicated by the arrows in Fig. 307.

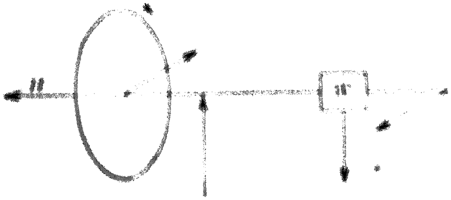


Fig. 307.

Now if a vertical force is applied to the end of the arm, say by pressing down on it with a lead pencil, the external moment is increased and hence the effect is to hasten the precession. If, however, the arm is pressed upward from below, the external moment is decreased and consequently the precession is retarded.

If the arm is pushed from the side in the direction in which it

is precessing, it will drop vertically, and if pushed from the other side so that the precession is retarded, it will rise.

With a large instrument of this kind the effect is so strong and tangible that it affords a convincing demonstration of the principle in question.

166. Kinetic Reactions. — As an illustration of the kinetic reactions which may arise from gyroscopic action, consider the case of a locomotive rounding a curve. The angular momentum of the wheels have the same direction as the axle. Consequently when the engine passes around a curve the direction of the angular momentum is changed, this change in momentum giving rise to gyroscopic forces tending to increase the pressure on one bearing and decrease it on the other.

It was shown above that to change the angular momentum H by an amount dH requires the action of an impulsive couple Mdt. Under the continued action of an impulsive couple Mdt the angular momentum H will be turned through an angle θ, and the succession of infinitesimal vectors $dH = Mdt$ will integrate into the finite impulsive couple Mt. Therefore, since an arc is equal to the product of its radius and the angle it subtends, we have (Fig. 308)

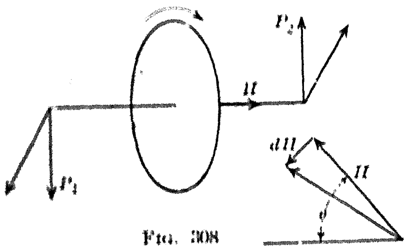


FIG. 308

$$Mt = H\theta = I\omega\theta.$$

If, then, Ω denotes the angular velocity of the engine about the curve, that is, if θ is the angle described in the time t, $\Omega = \frac{\theta}{t}$, and consequently

$$M = I\omega\frac{\theta}{t} = I\omega\Omega.$$

Now if l denotes the distance between centers of bearings and F, F the kinetic reactions on same, then

$$M = Fl,$$

and hence

$$F = \frac{I\omega\Omega}{l}.$$

Let v denote the linear speed of the train, r the radius of the wheels, and R the radius of the curve. Then

$$\omega = \frac{v}{r}, \quad \Omega = \frac{v}{R}$$

and consequently

$$F = \frac{Iv^2}{Rr}$$

PROBLEM

420. The radius of gyration of the wheels of a locomotive is [illegible] ft., the diameter of wheel approximately [illegible], the distance between centers of bearings is [illegible] ft., the speed on the curve [illegible], and the curve around which it is passing has a radius of 600 ft. Compute the [illegible] reaction [illegible].

SOLUTION. The moment of inertia [illegible] is the sum of the moments of inertia for both wheels [illegible]. Hence F [illegible] W approximately.

167. Ordinary Top. Suppose that an ordinary top is spun in an inclined position, as shown in Fig. 309. The weight of the top will then have a tendency to cause it to fall, and thereby produce a change in the angular momentum. Let ΔH denote this change in angular momentum, and resolve ΔH into horizontal and vertical components. The horizontal component has a tendency to cause the top to tip over in a direction at right angles to that in which it leans, whereas the vertical component produces rotation about the vertical axis AB, that is, causes precession. As soon as the top begins to precess, however, the change in the angular momentum H becomes horizontal instead of vertical. This calls forth a resisting moment Wr, which just balances the moment due to its weight.

Fig. 309

To find the rate of precession and its period let Ω denote the angular velocity of precession, and a the distance from the center of gravity of the top to the point of support B (Fig. 310). From

the preceding article we have $Mt = I\omega\theta$. Consequently $I\omega\Omega = M = Ws\sin\beta = Wr$.

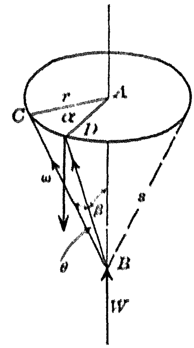


FIG. 310

Moreover, in the time t the center of gravity of the top describes an arc $DC' = \frac{Mt}{I}$. Hence, the angular displacement α is found to be

$$\alpha = \frac{\text{arc}}{\text{radius}} = \frac{\frac{Mt}{I}}{\omega\sin\beta} = \frac{Wst\sin\beta}{I\omega\sin\beta} = \frac{Wst}{I\omega}.$$

Therefore, since the velocity of precession is constant, we have for the period P,

$$P = \frac{2\pi t}{\alpha} = \frac{2\pi I\omega}{Ws}.$$

From this relation it is evident that the precession is slower the greater the angular velocity ω, and also that the velocity of precession is independent of the inclination β.

By observing the period of precession P, the angular velocity of spin ω may be determined.

168. Effect of Friction. — It is well known that when an ordinary top is spun on a rough surface, it will gradually rise until its axis is vertical, and then remain in this position, when it is said to "sleep." This is due to the effect of friction on the point of the top, or peg. If the surface on which the top spins is rough, it will experience a frictional resistance tangential to the peg. Let this be denoted by F, that is, $F = \mu W$, where μ denotes the coefficient of friction. The moment of this frictional resistance F about the center of gravity C produces a couple at right angles to the vector momentum H, as indicated by the vector M, in Fig. 311. The resultant H' of the two vectors H and Mdt is more nearly vertical than H, and hence there is a constant tendency for H to become vertical, that is, for the top to rise.


FIG. 311

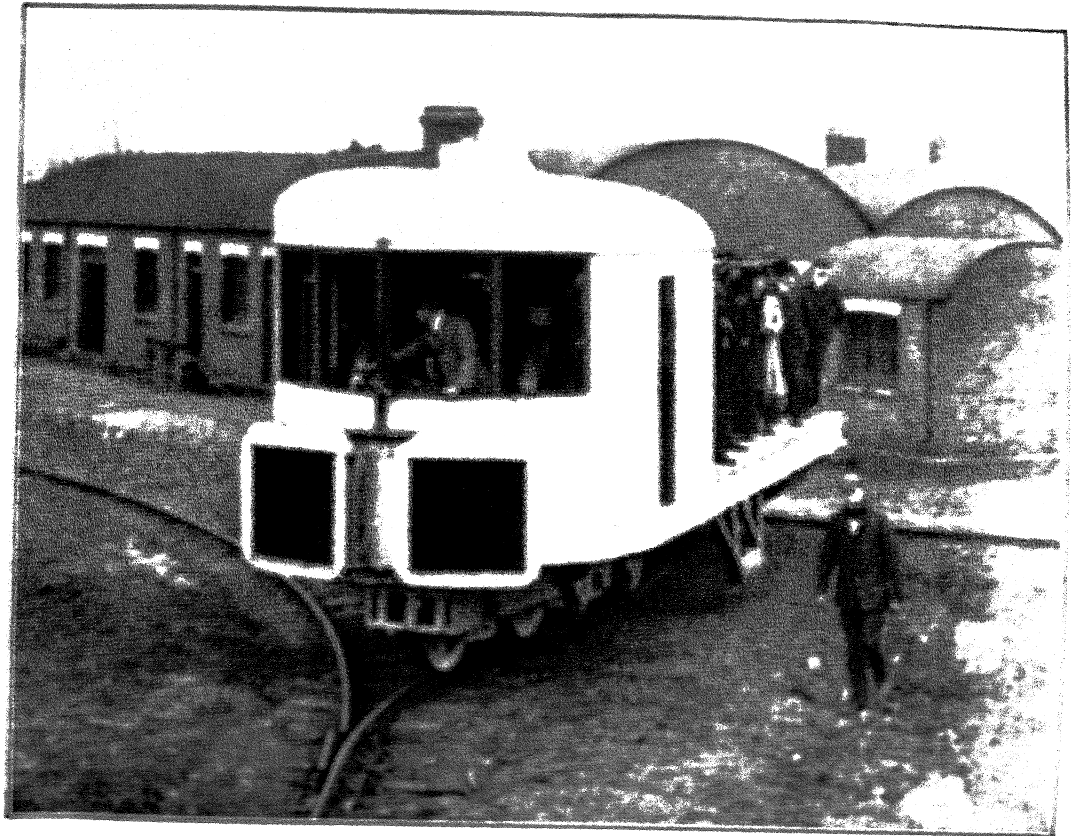


FIG. [illegible] — Front view of the Brennan monorail car

The platform of the car was ten feet wide by forty feet long, with a steel cab for the engineer on its forward end. During the trial trips it carried forty passengers

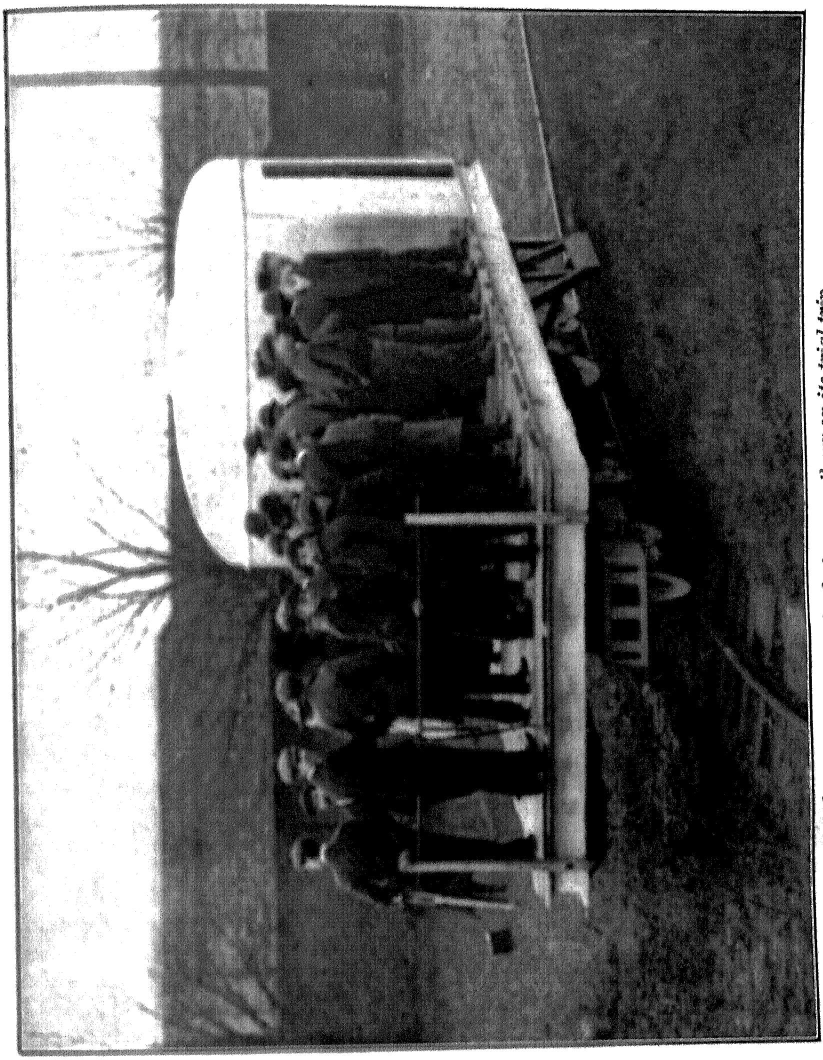


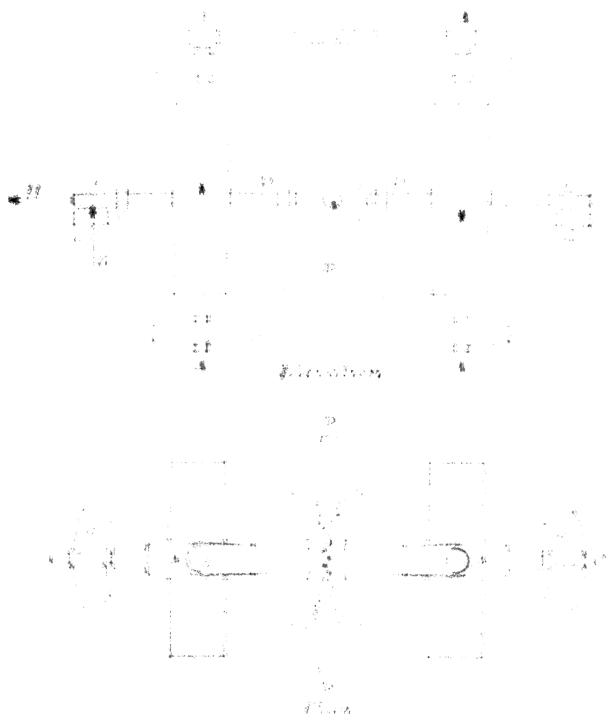
FIG. 312 *b*.—*The first monorail car on its trial trip*

Showing the car taking a curve while unevenly loaded with passengers. The equilibrium was perfectly maintained by means of two gyroscopes weighing three fourths of a ton each, and making three thousand revolutions a minute.

The smoother the surface on which the top spins, the less will be its tendency to rise.

169. Monorail Car. A notable application of the effect of friction, explained in the preceding article, is that made use of in the Brennan monorail car. This consists of a car platform mounted on a pair of two-wheeled bogie trucks, and equipped with a pair of heavy gyroscopes revolving at high speed, by reason of which it is possible for the car to remain upright on a single rail, as shown in Fig. 312 (see pages 422 and 423).

The apparatus used for balancing the car is shown in outline in Fig. 313. Without going into the details of construction,



Elevation

Plan

FIG. 313.

this consists essentially of a pair of gyroscopes, each wheel weighing ¾ ton, and revolving at a speed of 3000 r.p.m., mounted in a double set of gimbals. The gimbal frame is suspended from a horizontal shaft HH, and each gimbal ring turns about a vertical

axis AA, these vertical axes being connected by toothed sectors, as shown in Fig. 313, so that a revolution of one gimbal ring about its vertical axis AA causes an equal and opposite revolution of the other gimbal ring about its vertical axis. The flywheels being revolved in opposite directions, as indicated in Fig. 313, the purpose of the sectors is to balance the constraint exerted by one gyroscope when the car is rounding a curve against that exerted by the other, so as to preserve the alignment of the entire apparatus with the car body.

To keep the car upright, the outer end of each flywheel axis C is made to project over a bearing surface G, attached to the side of the car. When the car is in perfect balance, these shafts C clear the bearing surfaces G by a small amount. Any tendency to tip brings one of the bearings G in contact with its shaft C, thereby creating a frictional resistance. This acts in the same manner as the frictional resistance on the peg of a top. Thus, if R denotes the pressure between bearing and shaft due to tipping of the car, a horizontal frictional force is developed of amount $F = \mu R$, which tends to cause horizontal precession. As soon as precession begins, however, the vector momentum H undergoes a vector change of amount, $dH = Mdt$, the direction of which is horizontal (Fig. 314). The impulsive couple Mdt is, therefore, also horizontal, that is to say, the forces constituting the couple M are vertical, pressing the shaft down on the bearing to the same extent as the upward reaction due to its lack of balance. Since the kinetic reaction or impulsive couple M depends on the frictional resistance μR, it is evident that this kinetic reaction adjusts itself automatically to the upward static reaction R.*

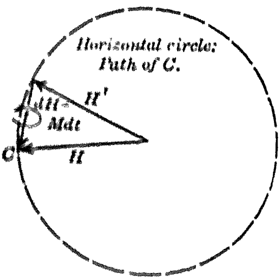


FIG. 314

170. Automatic Torpedo. — Another commercial application of gyroscopic action is found in the automobile torpedo invented by

* For a more detailed description of the monorail car, see *Engineering*, London, Vol. 83, pp. 625, 749, 794. For a popular description see *McClure's Magazine*, February, 1910, or *Cassier's Magazine*, February, 1910.

Admiral Howell of the United States Navy. The motive power for the torpedo is furnished by a small flywheel weighing 135 lb., and revolving at 10,000 r. p. m. This wheel is mounted on a horizontal axis, so that the vector momentum H is perpendicular to the direction in which the torpedo moves, as shown in Fig. 315.

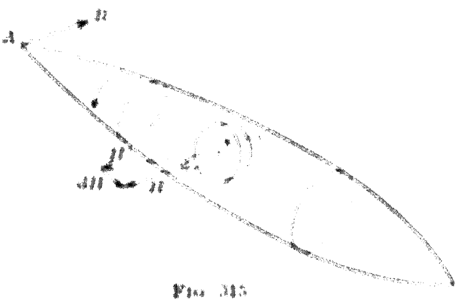


Fig. 315

Now suppose that a horizontal force AB acts so as to turn the torpedo out of its course. Instead of moving in the direction AB, the gyroscopic forces developed will cause it to roll in the direction indicated by the curved arrow. That is to say, the horizontal force AB will tend to rotate H in a horizontal plane, and produce a vector change, $\Delta H = M\Delta t$, perpendicular to H, or parallel to the direction in which the torpedo is moving. The forces of the couple M must then be such as to produce rolling in the direction shown by the curved arrow, namely, the direction in which a screw must be turned to advance in the direction indicated by ΔH.

This rolling action is utilized by means of a pendulum within the torpedo, so arranged as to bring into action a set of rudders which steer the torpedo back into its original course.

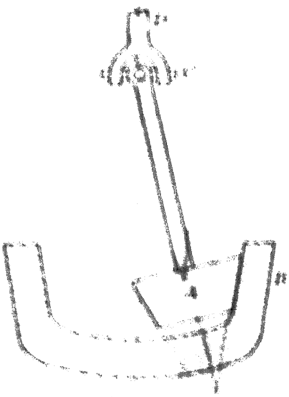


Fig. 316

171. Griffin Grinding Mill. Another successful commercial application of gyroscopic motion is the Griffin grinding mill. This consists of a heavy roller or pestle A (Fig. 316) mounted on a shaft which is hung from a universal joint C, and rotated by means of a motor geared to the shaft at D. This roller revolves inside a steel hopper B. When the shaft is set in rotation, the axis hangs vertically, and shows no tendency to change its direction. If, however, the roller A is swung sidewise until it touches the side of the hopper B, it immediately presses

against it with great force, rolling around on the inside of the pan at a high velocity.

To explain this phenomenon, let C_1 (Fig. 317) represent a circular section of the roller A, and C_2 the circular path, or section, of the hopper B, on which it rolls. Then, in passing from one position to a consecutive position, the angular momentum H is changed by an amount $dH = M\,dt$. This vector change dH is in a direction tangential to the circular path C_2. Hence, the moment M, which has the same direction as dH, must consist of a couple whose forces are perpendicular to H, that is to say, normal to the circles at their point of contact. One of these forces is the reaction of the bearing at C, and the other is the normal pressure between the surfaces in contact, on which the mechanism depends for its effectiveness.

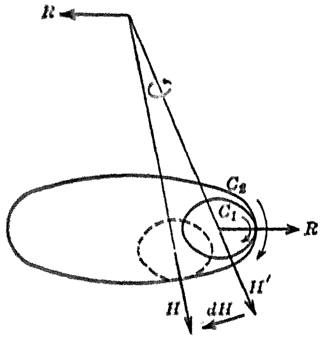


FIG. 317

172. Rolling Cones. In the illustrations of gyroscopic motion considered above, there are in each case two motions: a rotation of the body about its axis, and a precession of this axis. The motion may therefore be described as a rotation of the body about an instantaneous axis, which is itself revolving about a fixed line. The revolution of the instantaneous axis of spin is what is called the precession. Gyroscopic motion thus affords an illustration of the general theorem stated in Art. 30, Chap. I, that any motion may be considered as made up of rotations about definite axes.

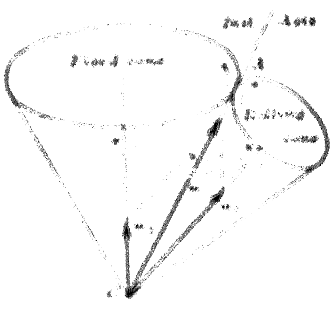


FIG. 318

The motion here considered has been illustrated geometrically by Poinsot by means of a pair of cones, one of which is fixed and the other rolling upon it. The motion of the Griffin grinding mill is a good illustration of what is meant.

The rolling cone in Poinsot's motion may be either internal or

external to the fixed cone. In Fig. 318 the rolling cone is shown external to the fixed cone, their line of contact at any instant being the instantaneous axis. This moves around the fixed cone, or precesses, in the same direction as that in which the body rotates, and in this case the motion is said to be **progressive**. The motion of an ordinary top is an example of such progressive precession.

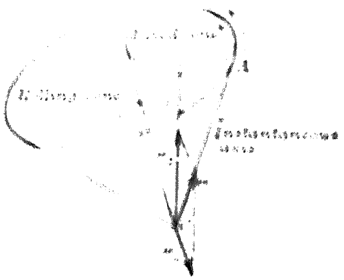


Fig. 318

The same relation exists if, instead of being convex, the cones are concave toward each other, as shown in Fig. 319. In both these figures the rolling cone is external to the fixed cone.

If the rolling cone is internal to the fixed cone, the instantaneous axis moves in the opposite direction to that in which the body rotates, and the precession is called **retrograde** or **regressive** (Fig. 320). An example of retrograde precession is that furnished by the motion of the earth, which is considered more fully in the next article.

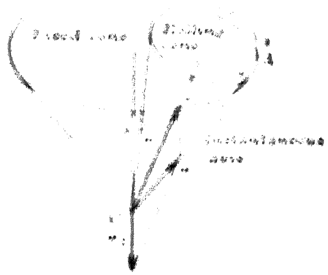


Fig. 320

To obtain an expression for the angular velocity of the body about the instantaneous axis, pass a plane perpendicular to this axis, and let r_1, r_2 denote the radii of the plane sections of the cones cut out by this plane at their point of contact (Fig. 321). Now suppose that one cone rolls on the other so that C_2 moves to C_2', and the point of contact passes from A to B. Let ds denote the length of the arc AB, $d\alpha$ the angle subtended by it at the center of curvature of one

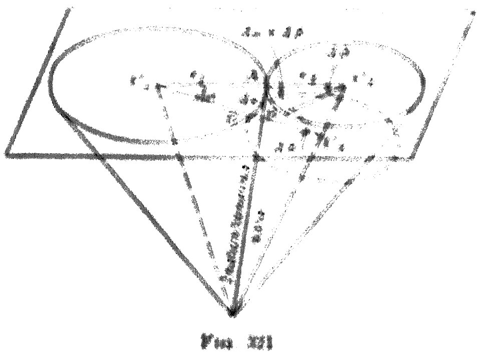


Fig. 321

section and $d\beta$ at the other. Then the angle turned through by the body itself in rolling over the arc ds is $d\alpha + d\beta$, and consequently its angular velocity ω about the instantaneous axis is

$$\omega = \frac{d\alpha + d\beta}{dt}.$$

Since $ds = r_1 d\alpha = r_2 d\beta$, this may be written

$$\omega = \frac{ds}{dt}\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Now let ω_1 and ω_2 denote the angular velocities with which the instantaneous axis is traveling about the axes of the two cones. Then if γ and δ denote the angles which these axes make with the instantaneous axis (Fig. 322), and p_1, p_2 denote the perpendiculars from A on these axes, we have

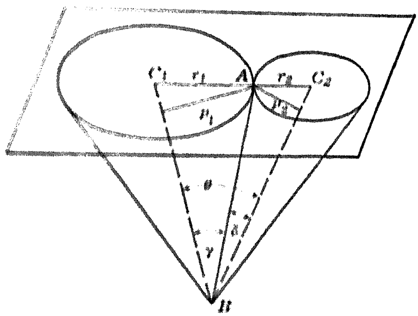


FIG. 322

$$\frac{ds}{dt} = p_1\omega_1 = r_1 \cos\gamma\, \omega_1,$$

and also

$$\frac{ds}{dt} = p_2\omega_2 = r_2 \cos\delta\, \omega_2.$$

Consequently

$$\frac{1}{r_1}\frac{ds}{dt} = \omega_1 \cos\gamma, \quad \frac{1}{r_2}\frac{ds}{dt} = \omega_2 \cos\delta,$$

whence by addition

$$\frac{ds}{dt}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \omega_1 \cos\gamma + \omega_2 \cos\delta.$$

Since from the equation derived above the left member of this expression is equal to ω, we have

$$\omega = \omega_1 \cos\gamma + \omega_2 \cos\delta,$$

and ω is therefore the vector resultant of ω_1 and ω_2, as shown in Fig. 323.

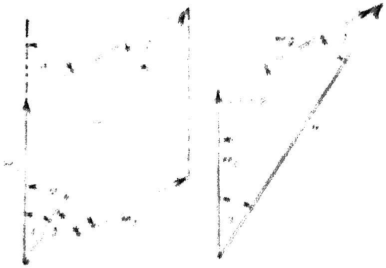


FIG. 323

Also from the vector triangle in Fig. 323, we have, by the law of sines,

$$\frac{\omega_1}{\sin\delta} = \frac{\omega_2}{\sin\gamma} = \frac{\omega}{\sin\theta},$$

and by the law of cosines

$$\omega^2 = \omega_1^2 + \omega_2^2 + 2\omega_1\omega_2\cos\theta,$$

from which the direction and amount of ω may be determined when ω_1 and ω_2 are known.

173. Precession of the Earth. — If both the fixed and rolling cones are cones of revolution, and ω_1, ω_2, and ω are all constant, the precession is called **regular**. An important case of regular precession is that furnished by the motion of the earth. Here the axis of the earth describes a cone of angle $\theta = 23° 27' 32''$ once in 25,868 years, with a retrograde motion, illustrated in Fig. 320. Therefore, taking one day as the unit of time, we have

$$\omega_1 = 2\pi \div 1, \quad \omega_2 = 2\pi \div 25868 \times 365.256,$$

and hence from the relation $\frac{\omega_1}{\sin\delta} = \frac{\omega_2}{\sin\gamma}$, we have

$$\sin\delta = \sin\gamma\,\frac{\omega_1}{\omega_2} = \frac{1}{25868 \times 365.256}\sin 23° 27' 32'',$$

whence

$$\delta = 0.0087''.$$

This angle is too small to affect astronomical measurements, the radius of the circle cut by this cone on the surface of the earth being only 10.63 inches.

174. Rotation about Principal Axes. — Consider the general expressions for the motion of a rigid body. If L_x, L_y, L_z denote the rectangular components of its linear momentum, and H_x, H_y, H_z the rectangular components of its angular momentum (Fig. 324),

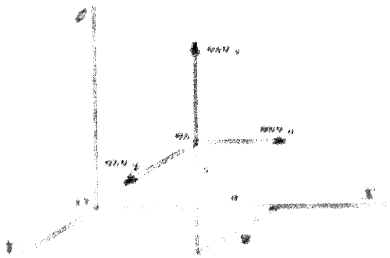


FIG. 324

the motion of the body is defined by the relations:

$$L_x = \sum m v_x, \quad H_x = \sum m(y v_z - z v_y),$$

$$L_y = \sum m v_y, \quad H_y = \sum m(z v_x - x v_z),$$

$$L_z = \sum m v_z, \quad H_z = \sum m(x v_y - y v_x).$$

Certain relations also exist between the linear and angular velocities of the system. Thus, referring to Fig. 325, if v_x, v_y, v_z denote the components of the linear velocity of a point and ω_x, ω_y, ω_z the components of its corresponding angular velocity, then forming relations similar to that deduced in Art. 22, Chapter I, we have

$$v_x = z\omega_y - y\omega_z,$$

$$v_y = x\omega_z - z\omega_x,$$

$$v_z = y\omega_x - x\omega_y.$$

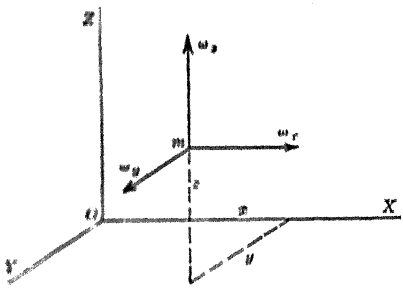


Fig. 325

Substituting these expressions for v_x, v_y, v_z in the equations for H_x, H_y, H_z given above, the result is

$$H_x = \sum (my^2\omega_x - mxy\omega_y - mxz\omega_z + mz^2\omega_x),$$

$$H_y = \sum (mz^2\omega_y - myz\omega_z - mxy\omega_x + mx^2\omega_y),$$

$$H_z = \sum (mx^2\omega_z - mxz\omega_x - myz\omega_y + my^2\omega_z).$$

In Art. 122, however, the moments and products of inertia of any system with respect to the coördinate axes were defined as follows:

$$A = \sum m(y^2 + z^2), \quad D = \sum myz,$$

$$B = \sum m(x^2 + z^2), \quad E = \sum mxz,$$

$$C = \sum m(x^2 + y^2), \quad F = \sum mxy.$$

In terms of these quantities the expressions for the components of the angular momentum become

$$H_x = A\omega_x - F\omega_y - E\omega_z,$$
$$H_y = -F\omega_x + B\omega_y - D\omega_z,$$
$$H_z = -E\omega_x - D\omega_y + C\omega_z.$$

To pass from the kinematical to the dynamical aspect of the motion, let X, Y, Z denote the components of the force acting at any point of the system, and M_x, M_y, M_z the components of the torque with respect to the origin. Then, as shown in Art. 63, Chapter II,

$$\frac{dI_x}{dt} = F_x, \quad \frac{dM_x}{dt} = H_x,$$
$$\frac{dI_y}{dt} = F_y, \quad \frac{dM_y}{dt} = H_y,$$
$$\frac{dI_z}{dt} = F_z, \quad \frac{dM_z}{dt} = H_z.$$

In general, any motion of a body gives rise to inertia forces which tend to alter the position of the instantaneous axis as the motion proceeds. Thus suppose that the instantaneous axis at any instant is taken for the z axis. Then if r denotes the distance of any point from this axis, and ω its angular velocity with respect to the axis, the centrifugal acceleration is $r\omega^2$ and the centrifugal force is $\sum mr\omega^2$, the rectangular components of which are

$$F_x = \sum mx\omega^2 = \omega^2 \sum mx = \omega^2 M\bar{x},$$
$$F_y = \sum my\omega^2 = \omega^2 \sum my = \omega^2 M\bar{y},$$
$$F_z = 0,$$

where M denotes the mass of the entire system and $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coördinates of its center of gravity. Moreover, the components of the torque or centrifugal couple C are, in this case (Fig. 326),

$$C_x = yF_z - zF_y = -\sum myz\omega^2 = -D\omega^2,$$
$$C_y = zF_x - xF_z = \sum mxz\omega^2 = E\omega^2,$$
$$C_z = xF_y - yF_x = 0.$$

The centrifugal couple tends in general to alter the position of the instantaneous axis. Suppose, however, that this axis passes through the center of mass. Then $x = y = z = 0$, and consequently the centrifugal force disappears. Moreover, if the instantaneous axis is a principal axis, the products of inertia $D = E = 0$, and the centrifugal couple also disappears. Hence

If a body is rotating about a principal axis, it will continue to rotate about this axis, permanently unless acted on by external forces.

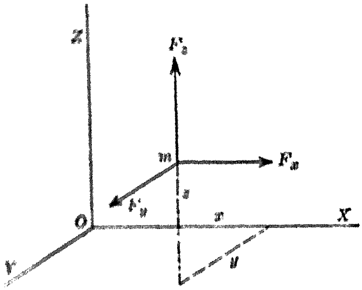


FIG. 326

This property of a principal axis, namely, that when used as an axis of rotation it will maintain its direction in space, is illustrated in all the cases of gyroscopic motion previously considered, since in each case the axis of spin was an axis of symmetry and therefore a principal axis.

Another notable application of this principle is found in Foucault's gyroscope, the axis of which points in a fixed direction in space while the earth turns, thus making apparent the motion of the earth, and enabling its rate to be measured experimentally.

173. Rotation referred to Poinsot's Ellipsoid. — If the instantaneous axis is not a principal axis, the centrifugal couple generates an angular momentum which compounds with the momentum already possessed by the body and thus alters the axis of spin.

Instead of taking the instantaneous axis for the Z-axis, as in the preceding article, take the three principal axes of the body for axes of coordinates, with origin at the center of mass. Since the products of inertia D, E, F all vanish in this case, the components of the angular momentum become simply

$$H_x = A\omega_x, \quad H_y = B\omega_y, \quad H_z = C\omega_z.$$

From Art. 122 the equation of the inertia ellipsoid, or Poinsot's central ellipsoid, is $$Ax^2 + By^2 + Cz^2 = 1.$$

Now let OP (Fig. 327) denote the direction of the instantaneous axis at any instant, and r the length of the radius vector OP

measured in this direction. Then, if x, y, z denote the coordinates of P, we have

$$\frac{\omega_x}{x} = \frac{\omega_y}{y} = \frac{\omega_z}{z} = \text{constant, say } k.$$

Hence, the components of the angular momentum may be written

$$H_x = Axk, \quad H_y = Byk, \quad H_z = Czk.$$

Suppose, now, that a tangent plane is drawn to the ellipsoid at the point P. Then, since the equation of the ellipsoid is $Ax^2 + By^2 + Cz^2 = 1$, the direction cosines of the tangent plane at any point x, y, z are proportional to Ax, By, Cz. But from what precedes, the components of the angular momentum are also proportional to Ax, By, Cz. Hence, the vector momentum H must be normal to this tangent plane, as shown by the line OQ in Fig. 327, drawn from the center of the ellipsoid perpendicular to the tangent plane through P.

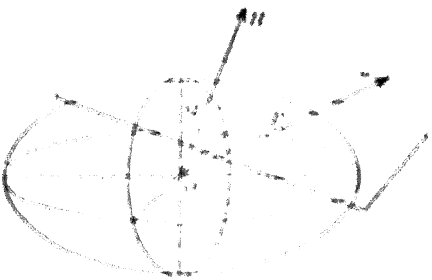


FIG. 327.

Conversely, if the angular momentum H is given, pass a plane through the center of the ellipsoid O perpendicular to H. Then the instantaneous axis will be that diameter of the ellipsoid which is conjugate to this diametral plane.

It is thus evident, geometrically, that when the instantaneous axis is a principal axis, the tangent plane drawn at its point of intersection with the ellipsoid will be perpendicular to it, and consequently H and ω will coincide. In this case, the instantaneous axis becomes a permanent axis, as otherwise explained in the preceding article.

176. Polhode and Herpolhode. In Art. 172 it was shown that when the instantaneous axis is taken for the Z-axis the components of the angular momentum become

$$H_x = -E\omega, \quad H_y = -D\omega, \quad H_z = C\omega,$$

and the components of the centrifugal couple are

$$C'_x = -D\omega^2, \quad C'_y = E\omega^2, \quad C'_z = 0.$$

Since C'_z is zero, the axis of the centrifugal couple must be perpendicular to the instantaneous axis. Moreover, since the relation

$$H_x C'_x + H_y C'_y + H_z C'_z = 0$$

is satisfied identically, the centrifugal couple is also perpendicular to the vector momentum H. Therefore, since the centrifugal couple is perpendicular to both of these lines, it is perpendicular to their plane.

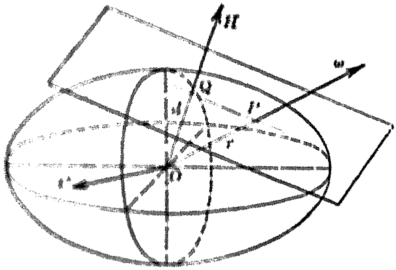


FIG. 328

Referring to Fig. 328, if the centrifugal couple C' is perpendicular to the plane of H and ω, it has no component parallel to this plane. Consequently, the component of ω in the direction H is unchanged by the centrifugal couple, that is,

$$\omega \cos(\omega H) = \text{constant}.$$

Since the vector angular momentum H is also constant in magnitude and direction, the product

$$H\omega \cos(\omega H) = \text{constant}.$$

But $H \cos(\omega H)$ is that component of the angular momentum which is parallel to the instantaneous axis, and is equal to the product of the angular velocity ω by the moment of inertia I of the body with respect to this axis, that is,

$$H \cos(\omega H) = I\omega.$$

Hence substituting this value in the above expression, the result is

$$\omega H \cos(\omega H) = I\omega^2 = \text{constant},$$

which is simply a statement of the principle of the conservation of energy.

Now if r denotes the radius vector OP (Fig. 325), we have from Art. 122

$$I = \frac{1}{r^2}$$

and consequently from the preceding article

$$I\omega^2 = \frac{\omega^2}{r^2} = k^2$$

Since $I\omega^2$ is constant, k is also constant. That is to say, the angular velocity ω is proportional to the radius vector r drawn to the ellipsoid of inertia in the direction of the instantaneous axis.

Let d denote the perpendicular distance OQ from O to the tangent plane at P. Then, denoting the angle between the vectors ω and H by the symbol (ωH), since $\omega \cos (\omega H) =$ constant, we must also have $r \cos (rd) =$ constant. But $r \cos (rd) = d$, and therefore $d =$ constant. Moreover, since d has the same direction as H, its direction in space is also constant. Therefore, since both the length and direction of d are constant, the tangent plane to which it is normal must be a fixed plane in space.

As the point of contact of this tangent plane with the ellipsoid of inertia is on the instantaneous axis, the ellipsoid must be turning about this radius vector and hence rolling without sliding on the fixed tangent plane. The motion may therefore be described as the rolling of an ellipsoid, whose center is fixed, on a fixed tangent plane, together with the condition that the angular velocity is proportional to the radius vector drawn to the point of contact of the ellipsoid with the plane.

The locus of the pole P of the instantaneous axis on the ellipsoid was called by Poinsot the **polhode** (πόλος, axis; ὁδός, path), and its locus on the tangent plane the **herpolhode** (ἕρπειν, to creep like a snake).

The distance d of the tangent plane at the point $P(x, y, z)$ from the center of the ellipsoid is

$$d = \frac{1}{\sqrt{A^2x^2 + B^2y^2 + C^2z^2}}$$

Since this is constant, its equation, together with that of the ellipsoid

$$Ax^2 + By^2 + Cz^2 = 1$$

defines the polhode. Writing the expression for d in the form

$$d^{-2}(A^2x^2 + B^2y^2 + C^2z^2) = 1,$$

and subtracting this from the equation of the ellipsoid, the result is

$$(A - A^2/d^2)x^2 + (B - B^2/d^2)y^2 + (C - C^2/d^2)z^2 = 0,$$

which represents a cone with vertex at O, which intersects the ellipsoid in the polhode curve. This is in fact the polhode or rolling cone, explained in Art. 170. From this method of defining the polhode as the intersection of an ellipsoid with a quadric cone, it is evident that the polhode curve is a twisted quartic.

INDEX